1st sheet of tutorial problems in special relativity, PHY206

1. A muon, rest mass $106\,\mathrm{MeV/c^2}$, has a kinetic energy of $3\,\mathrm{MeV}$ in some reference frame.
(a) Calculate γ for the muon.
(b) What regime is the muon in?
Using the rules for the different regimes, make suitable approximations in the following:
(c) Calculate β for the muon.
(d) Check for consistency with the chosen regime.
(e) Calculate the muon momentum, in MeV/c.

2. Repeat question 1 for the following particles:
(a) A proton, mass $938\,\mathrm{MeV/c^2}$, having a total energy of $4\,\mathrm{GeV}$.
(b) An α–particle, mass $3.73\,\mathrm{GeV/c^2}$, having a total energy of $3.83\,\mathrm{GeV}$.
(c) An axion, mass $3.5\,\mu\mathrm{eV/c^2}$, having a total energy of $3\,\mathrm{keV}$.

3. A neutron collides with a nucleus of boron-10, $^{10}\mathrm{B}$. If they are approaching from opposite directions and collide head-on, what does β for the $^{10}\mathrm{B}$ nucleus have to be for it to have equal and opposite momentum to the neutron, if the kinetic energy of the neutron is $14.4\,\mathrm{MeV}$? The rest mass of $^{10}\mathrm{B}$ is $9.33\,\mathrm{GeV/c^2}$, and the neutron rest mass is $940\,\mathrm{MeV/c^2}$.

Solutions to tutorial problems in special relativity, PHY206

First, note that I regard learning to use electron volt units as a key learning outcome of this course. I have told the students in about 18 different ways that they should NOT convert to SI units, and should instead learn to do these problems with velocities expressed as a fraction of the speed of light, so that $\beta = v/c$, and $v = 0.3c$ means $\beta = 0.3$, not $v = 9 \times 10^7\,\mathrm{m\,s^{-1}}$. Please help me by teaching the students to do these problems in tutorials using the approach below. Thanks!

1(a) if T is the kinetic energy, E is the total energy, and $E_R = m_0c^2$ is the energy at rest, then $T = (\gamma - 1)m_0c^2 = (\gamma - 1)E_R$. Therefore $\gamma = T/E_R + 1$, which in this case is 1.03.

1(b) I have taught the students to think in terms of three regimes, non-relativistic, mildly relativistic, and highly relativistic. These three regimes are characterised by different ranges of β and γ. The diagram below is hopefully helpful. It's a plot of γ as a function of β for $0 < \beta < 1$. You can see from this plot that the muon is in the non–relativistic regime.

1(c) Because we're in the non–relativistic regime, we can use the non–relativistic approximation for kinetic energy, $T \simeq 1/2m_0v^2$, but in the form $T \simeq 1/2(m_0c^2)(v/c)^2 = E_R\beta^2/2$. Therefore $\beta = \sqrt{2T/E_R} = \sqrt{2 \times 3[\mathrm{MeV}]/106[\mathrm{MeV}]} = 0.23$.

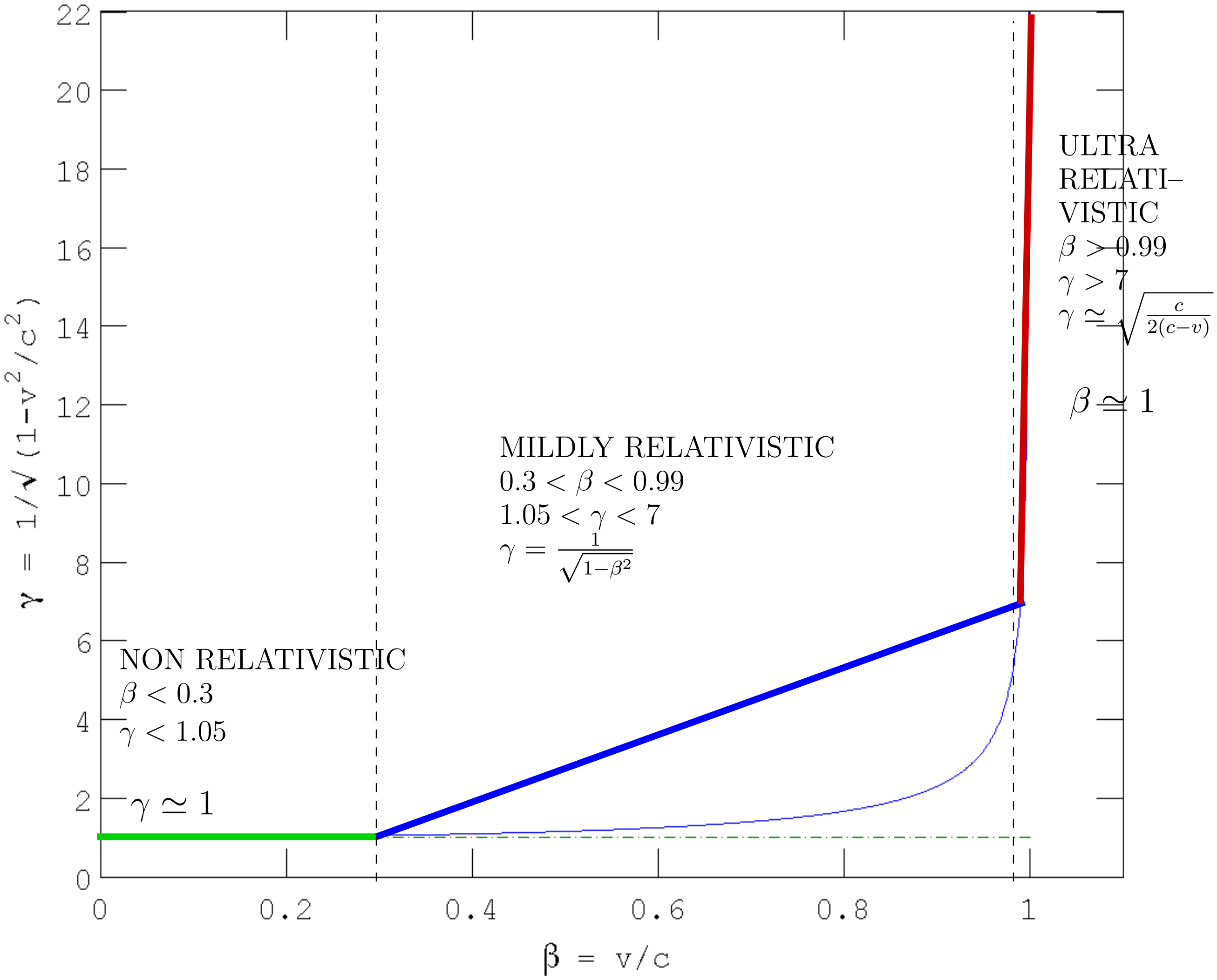
1(d) $\beta = 0.23$ is consistent with the non–relativistic regime.

1(e) Again in the non–relativistic regime, $T \simeq p^2/2m_0 = (pc)^2/2(m_0c^2) = (pc)^2/2E_R$, so that $pc \simeq \sqrt{2E_RT}$, and if E_R and T are in MeV, then $p[\mathrm{MeV/c}] \simeq \sqrt{2E_RT}[\mathrm{MeV}]/\mathrm{c}$. In this case the momentum is $\sqrt{2 \times 106[\mathrm{MeV}] \times 3[\mathrm{MeV}]}/c = 25.2\,\mathrm{MeV/c}$.

2(a) The total energy $E = \gamma m_0c^2 = \gamma E_R$, so that $\gamma = E/E_R = 4[\mathrm{GeV}]/0.938[\mathrm{GeV}] = 4.26$. This is consistent with the mildly relativistic regime. In this case you must use $\gamma = 1/\sqrt{1-\beta^2}$, so that $\beta = \sqrt{1 + 1/4.26^2} = 0.972$. This is consistent with the mildly relativistic regime. The muon momentum must be calculated with $p = \gamma mv$, or $pc = \gamma(m_0c^2)(v/c)$, or $\mathrm{p[MeV/c]} = \gamma \mathrm{E_R}\beta[\mathrm{MeV}]/\mathrm{c}$, which is $p[\mathrm{MeV/c}] = 4.26 \times 938[\mathrm{MeV}] \times 0.972/\mathrm{c} = 3883\,\mathrm{MeV/c}$. You can check that $E^2 = (pc)^2 + (m_0c^2)^2$, so that $4000^2 = 3883^2 + 938^2$, but they haven't seen this formula since last year.

2(b) $\gamma = E/E_R = 3.83[\mathrm{GeV}]/3.73[\mathrm{GeV}] = 1.026$. This is therefore the non–relativistic regime. The kinetic energy T is $T = E - E_R = 0.1\,[\mathrm{GeV}]$. $\beta = \sqrt{2T/E_R} = 0.054$, consistent with the non–relativistic regime. The momentum is $p[\mathrm{MeV/c}] \simeq \sqrt{2E_RT}[\mathrm{MeV}]/\mathrm{c} = 863\,\mathrm{MeV/c}$. Again, you can check with $E^2 = (pc)^2 + (m_0c^2)^2$.

2(c) This time $\gamma = E/E_R = 3 \times 10^3[\mathrm{eV}]/3.5 \times 10^{-6}[\mathrm{eV}] = 8.6 \times 10^8$. This particle is highly relativistic. It's kinetic energy is approximately equal to its total energy, the mass energy being negligible. β is 1 because the mass can be neglected; it's behaving like a photon.



Again, this is consistent with highly relativistic regime. The momentum is given by $E \simeq pc$, so that $p[\text{keV/c}] \simeq E[\text{keV}]/\text{c} = 5\,\text{keV/c}$.

3. For the neutron, $T = (\gamma - 1)m_0c^2 = (\gamma - 1)E_R$, so that $\gamma = 1 + T/E_R = 1 + 14.4[\text{MeV}]/940[\text{MeV}] = 0.015$, so the neutron is in the non–relativistic regime. Its momentum is therefore $pc \simeq \sqrt{2TE_R}$, so that $p = \sqrt{2 \times 14.4 \times 940}\,\text{MeV/c} = 165\,\text{MeV/c}$. For the boron nucleus to have the same magnitude momentum, it will clearly also be in the non-relativistic regime, since it is a lot heavier than the neutron, so it will be moving a lot more slowly. So we can use $p \simeq m_0v$ so that $pc \simeq \beta m_0c^2 = \beta E_R$. Therefore for boron $\beta = pc/E_R = 165[\text{MeV}]/9.33[\text{GeV}] = 0.018$.