Lecture 4 - Lorentz Invariants

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1 Review of Lecture 3

Last time we studied the scattering of photons off particles. Defining E_γ^i as the energy of the γ–ray before scattering, E_γ^f as its energy after scattering, and θ as the scattering angle, we used conservation of energy and the three components of momentum from before the scattering to after it to show that

$$\frac{(E_\gamma^i - E_\gamma^f)}{E_\gamma^i E_\gamma^f} = \frac{1-\cos\theta}{m_0 c^2}. \tag{1}$$

Defining Δ to be the energy lost by the γ–ray, equal to $E_\gamma^i - E_\gamma^f$, we were also able to show that the maximum energy imparted to a particle of rest mass m is

$$\Delta = \frac{(E_\gamma^i)^2}{E_\gamma^i + \frac{mc^2}{2}}. \tag{2}$$

This lost energy is imparted to the massive particle. If that particle is an electron, it may interact in detectors designed to measure the γ–rays. Figure 1 shows a spectrum taken with a γ–ray spectrometer. Notice that there are two examples of narrow peaks at high energies, then a gap, then the edge of a broader, flattish spectrum of lower energy particles. These are Compton scattered electrons, and the highest energy they can have is δ given by the formula above. So given the energy of the top of the Compton spectrum, you can relate this energy Δ to the energy of the narrow peak, which is E_γ^i, using Equation 2.

2 Transforming between different inertial observers

So far we have emphasised conservation of energy and momentum, with the caveat that these quantities are only conserved if the same observer measures them before and after, this observer defining a single inertial (non–accelerating) frame of reference in which all measurements are made. However, we know how to transform between different inertial observers. It's time to review what we know about this and see how it can be extended, again with a view to making calculations easier.

First, we know how to transform the space–time coordinates of an event between their values measured by different observers having a relative velocity $v = \beta c$ between them aligned with the x–axis. Suppose the coordinates of an event for observer O are: (ct, x, y, z), and those for a second observer O' are (ct', x', y', z'). If O' is moving to the right along the x–axis with respect to O at velocity $v = \beta c$, then the coordinates are related by the Lorentz transformations, which are (of course),

$$
\begin{aligned}
ct' &= \gamma(ct - \beta\, x) \\
x' &= \gamma(x - \beta\, ct) \\
y' &= y \\
z' &= z
\end{aligned}
\tag{3}
$$

We also learned that another set of four quantities have exactly the same coordinate transformations between their values for observers O and O'. These four quantities are $(E/c, p^x, p^y, p^z)$, the energy divided by c, and the three components of the momentum, of a particle. So we can write the Lorentz transformations on energy and momentum like this:

$$
\begin{aligned}
E'/c &= \gamma(E/c - \beta\, p^x) \\
p^{x'} &= \gamma(p^x - \beta\, E/c) \\
p^{y'} &= p^y \\
p^{z'} &= p^z
\end{aligned}
\tag{4}
$$

It turns out that the Lorentz transformations for energy and momentum are in many ways more useful than the Lorentz transformations for time and position, especially when you are doing particle physics problems. So let's stop and do an example to see how we can use these transformations.

An electron and a positron are approaching each other with equal and opposite momentum in the lab. The total energy of each particle in the lab is 45 GeV. Find the total energy of one proton in the rest frame of the other one.

To solve this one, pick a particle whose rest frame we are going to transform to. Say the electron is moving in the $+x$ direction and the positron is moving in the $-x$ direction. Let's transform the positrons energy into the lab frame of the electron. First, we write down the transformation equation for the energy. Let E be the energy of the positron in the lab, and E' be the energy of the positron in the rest frame of the electron. Next, identify the regime the electron and positron are in. They are highly relativistic because $\gamma = E/(mc^2)$, which is 45GeV/0.511MeV=$88,000$, which is much bigger than 7, the lower limit for the highly relativistic regime. Next, we write down the Lorentz transformation for energy from Equation 4. We note that the primed frame is the rest frame of the positron, which is moving in the $+x$ direction with respect to the lab frame, hence the sign of β is correct in Equations 4.

$$E'/c = \gamma(E/c - \beta\, p^x). \tag{5}$$

. But because we're in the highly relativistic regime, we set $\beta = 1$ and obtain

$$E'/c = \gamma(E/c - p^x). \tag{6}$$

Now for the electron, because again it's highly relativistic, $E = cp$, where p is the magnitude of the momentum of the electron in the lab frame. The sign of the electron momentum is negative, so the momentum is $p^x = -E/c$. The only other thing we don't know is γ, which is the gamma factor for the positron in the lab frame, which is $88,000$ as calculated above. Putting numbers into Equation 6, we get

$$E'/c = \gamma(45[\text{GeV}]/c - (-45[\text{GeV}]/c)) \tag{7}$$

Multiplying all terms by c and cancelling the double minus sign to add up the two 45GeV terms to 90GeV, we obtain

$$E' = \gamma \times 90[\text{GeV}] = 88,000 \times 90[\text{GeV}] = 7.9 \times 10^6\,[\text{GeV}]. \tag{8}$$

Again, if we use energy units and are careful to make simplifying approximations, these calculations really aren't difficult. The main hazard with this one is to be clear about the sign of the momentum. Because we chose to transform to the frame of the right–going particle, there was already a $-$ sign by the β, but this was cancelled by the negative sign in the momentum of the positron, travelling to the left. Now, suppose we had chosen to transform to the rest frame of the positron from the lab frame. Then the Lorentz transformations would have had a + sign by the β, but the momentum of the electron in the lab is also positive. In both cases, you end up with 90GeV in the bracket, in one case because the two $-$ signs cancel, and in the other because there are no minus signs. So, everything works out.

3 Four vectors

Because (ct, x, y, z) and $(E/c, p^x, p^y, p^z)$ have the same transformations under changes of coordinate, we call them both 4–vectors. The vectors with which you are familiar can be defined as objects possessing a magnitude and a direction, or objects that transform in a well defined way under rotations of coordinate system. 4–vectors are defined as objects that transform under the Lorentz transformations when converting between the measurements made by two different inertial (non–accelerating) observers.

3.1 Four-displacement

An event at coordinates (ct, x, y, z) is represented by the 4–displacement x^ν, where ν is a number that can take values 0,1,2 or 3. The four components x^ν are $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. So far we have considered displacements from the origin only. What about small displacements between two neighbouring points. These too are four displacements, dx^ν. You can see this by writing out two copies of the Lorentz transform for neighbouring events, and subtracting them. So suppose we have two events at (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2), such that $ct_2 - ct_1 = c\,dt$, $x_2 - x_1 = dx$, $y_2 - y_1 = dy$, and $z_2 - z_1 = dz$. Then we could write for the time component of the Lorentz transform,

$$\begin{aligned} ct'_2 &= \gamma(ct_2 - \beta\, x_2) \\ ct'_1 &= \gamma(ct_1 - \beta\, x_1) \end{aligned} \tag{9}$$

and therefore, subtracting the two equations

$$c\,dt' = \gamma(c\,dt - \beta\,dx). \tag{10}$$

And we see that small displacements are also four–vectors, because the have the same transformation rules as the coordinates of events referred to the origin.

3.2 Four–momentum

Next, we define the four momentum of a particle as p^μ, where $p^0 = E/c$, $p^1 = p^x$, $p^2 = p^y$, $p^3 = p^z$. From Equation 4, p^μ is a four–vector.

4 Lorentz Invariants

We have seen that a class of objects called four–vectors transform in a known way, by the Lorentz transformations, between observers. Are there other objects that have known transformations between observers? Yes there most certainly are! Let's see if we've already met any. Recall the time dilation formula, relating the time interval $d\tau$ between two events in their rest frame, the frame where the two events occur in the same place, and the time interval dt' between the same two events in an inertial (non–accelerating) frame where the two events occur in different places. So, for example, $d\tau$ could be the time between two ticks on a clock at rest in the lab, hence occurring at the same place in the lab frame, and dt' could be the time interval between the same two ticks in a frame in which the clock is moving at a constant velocity. They are related by

$$dt' = \gamma\, d\tau = \frac{d\tau}{\sqrt{1-\beta^2}}. \tag{11}$$

Because γ is always greater than 1, the events always have a greater time interval between them in a frame in which the events do not occur in the same place. Hence, moving clocks run slow. Now, any observer is free to measure the speed of the moving clock and use this speed to calculate γ between the clocks rest frame and his rest frame, then use Equation 11 to calculate $d\tau$. If multiple, different, inertial observers all carry out this procedure, they will all get different answers for γ, but they will all get the same answer for $d\tau$. Therefore $d\tau$ is an invariant quantity, a quantity that is the same when calculated by all inertial observers. It is a an example of a *Lorentz invariant*.

Can we find some more Lorentz invariants? Yes we can. Recall the relationship between energy, momentum and rest mass of a particle

$$E^2 = p^2c^2 + m_0^2c^4. \tag{12}$$

Now, rearrange this formula, and we obtain

$$m_0c^2 = \sqrt{E^2 - p^2c^2} \tag{13}$$

Once again, some inertial observer sees a particle, and measures its total energy E and momentum magnitude p. This observer substitutes these numbers into Equation 13, and calculates the rest energy of the particle, m_0c^2. A different inertial observer gets different values E' and p' for energy and momentum, but the formula will give them the same value for the rest energy of the particle. Therefore the rest energy of the particle, m_0c^2 is a Lorentz invariant quantity. Any inertial observer calculating the rest energy will get the same answer.

5 Lorentz invariants from 4–vectors

There is a close relationship between Lorentz invariants and four vectors. It turns out that one can always calculate a Lorentz invariant from a four–vector, using the same procedure every time. The procedure is: suppose we have a four vector A^μ. This is any quantity that transforms under Lorentz transformations parallel to the x axis like dx^μ and p^μ:

$$\begin{aligned} A^{0'} &= \gamma A^0 - \beta\gamma A^1 \\ A^{1'} &= -\beta\gamma A^0 + \gamma A^1 \\ A^{2'} &= A^2 \\ A^{3'} &= A^3. \end{aligned} \tag{14}$$

To construct an invariant quantity out of the components of A^μ, we simply write down,

$$|A|^2 = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2. \tag{15}$$

It turns out, this combination is always Lorentz invariant, for any four–vector. You can prove it easily if you like, by substitution, but I'm not going to do it because I want to show you how to use it instead. Let's figure out the Lorentz invariants corresponding to dx^μ and p^α. For dx^μ we get

$$-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2 = -c^2\,dt^2+dx^2+dy^2+dz^2 = ds^2 \tag{16}$$

This is the so–called Lorentz invariant interval between two events. The fact that it is invariant follows directly from the time dilation formula of Equation 11, if you set $v\,dt' = dx'$, where dx' is the distance the clock moves in time dt'. Again, though, I'm not going to prove it now. For p^α we get

$$-(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -(\frac{E}{c})^2 + (p^x)^2 + (p^y)^2 + (p^z)^2. \tag{17}$$

Comparing with Equation 13 we can see that this sum is equal to $-m_0^2c^2$.

$$-(p^0)^2+(p^1)^2+(p^2)^2+(p^3)^2 = -(\frac{E}{c})^2+(p^x)^2+(p^y)^2+(p^z)^2 = -m_0^2c^2. \tag{18}$$

Because the result is equal to a product of the rest mass squared and $-c^2$, and because both the rest mass and c are observer independent, we see that we have again calculated an invariant quantity.

6 The centre of mass energy

We are now going to study another Lorentz invariant quantity, the centre of mass energy available in a collision. Let us consider HERA, an accelerator in which protons are accelerated up to a total energy of 920 GeV, and collided with electrons which have been accelerated up to a total energy of 27.6 GeV. The question is, what is the highest mass particle that could be created in such a collision? To find this out, we need to calculate the centre of mass energy of the accelerator. This is the total energy in the frame where the sum of the momenta of the particles is zero. Let's do this calculation using 4–vectors. Consider the incident particles in the lab frame. Both are in the highly relativistic regime. We have already written down the energies. The momentum of each proton, taking them to be moving in the $+x$ direction at the collision point, is +920 GeV/c. The momentum of each electron, taking them to be moving in the $-x$ direction, is -27.6 GeV/c Hence the sum of the energies is 947.6 GeV and the sum of the momenta is 892.4 GeV/c. Hence, Now we calculate the Lorentz invariant quantity,

$$-\frac{E^2}{c^2}+p^2 = -947.6^2[\mathrm{GeV}^2]/c^2+892.4^2[(\mathrm{GeV/c})^2] = -101568[\mathrm{GeV}^2/\mathrm{c}^2]. \tag{19}$$

Now, this quantity is equal to $-M_0^2c^2$. What is that? What is M_0? Remember that the whole quantity is Lorentz invariant, which means that it's the same in all frames. So, pick the centre of mass frame! In this frame, there is no momentum, only energy. Any energy in this frame is available to make new particles. And, the heaviest particle you could create is a single particle whose rest energy is equal to M_0c^2, as this would use up all the energy available, wasting nothing on kinetic energy. Consider, you can only do this in the centre of mass frame, because in any other frame there is momentum, and hence you have kinetic energy. Therefore,

$$-\frac{E^2}{c^2}+p^2 = -M_0^2c^2 = -101568[\mathrm{GeV}^2/\mathrm{c}^2]. \tag{20}$$

and

$$M_0c^2 = 318.7\,[\mathrm{GeV}]. \tag{21}$$

M_0c^2 is called the centre of mass energy, Sometimes it's given the symbol $\sqrt{s}$. So the heaviest particle you could possibly create in this accelerator would have an mass of $318.7\,\mathrm{GeV/c^2}$. Note that in reality this would never happen - the protons are not of course really point particles, so you could never really have a point-like collision between an electron and a whole proton; instead you would get a deep inelastic scattering event where the electron strikes a constituent quark or gluon of the proton. Indeed, this is how quarks were first discovered!

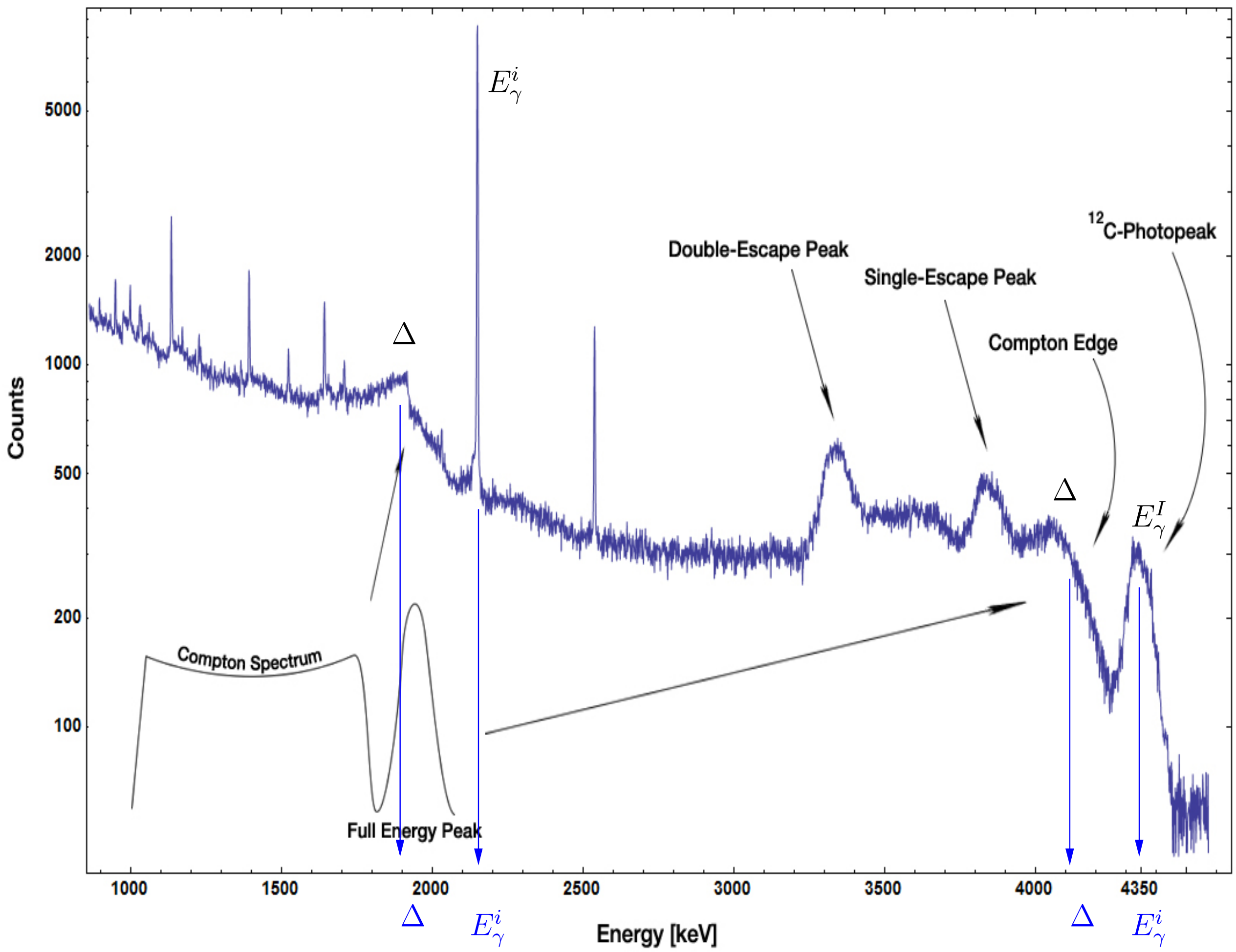

Figure 1: The energy spectrum of gamma rays from an AmBe source, as measured by a gamma ray spectrometer. Taken from `http://upload.wikimedia.org/wikipedia/commons/f/f2/Am-Be-SourceSpectrum.jpg`