Lecture 2 - Energy and Momentum

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1 Energy

In discussing energy in a relativistic course, we start by considering the behaviour of energy in the three regimes we worked with last time. In the first regime, the particle velocity v is much less than c, or more precisely $\beta < 0.3$. In this regime, the rest energy E_R that the particle has by virtue of its non–zero rest mass is much greater than the kinetic energy T which it has by virtue of its kinetic energy. The rest energy is given by Einstein's famous equation,

$$E_R = m_0 c^2 \tag{1}$$

So, here is an example. *An electron has a rest mass of* $0.511\,\mathrm{MeV/c^2}$. *What is it's rest energy?*.

The important thing here is to realise that there is no need to insert a factor of $(3 \times 10^8)^2$ to convert from rest mass in $\mathrm{MeV/c^2}$ to rest energy in MeV. The units are such that 0.511 is already an energy in MeV, and to get to a mass you would need to divide by c^2, so the rest mass is $(0.511\,\mathrm{MeV})/\mathrm{c^2}$, and all that is left to do is remove the brackets. If you divide by 9×10^{16} the answer is indeed a mass, but the units are $\mathrm{eV\,m^{-2}s^2}$, and I'm sure you will appreciate why these units are horrible. Enough said about that.

Now, what about kinetic energy? In the non–relativistic regime $\beta < 0.3$, the kinetic energy is significantly smaller than the rest energy. Suppose we are right at the edge of this regime, so $v = 0.3c$, or $\beta = 0.3$. What then is the kinetic energy? Well, the exact expression for kinetic energy is

$$T = (\gamma - 1)m_0c^2. \tag{2}$$

But, since we are in the non–relativistic limit, we may also expect that

$$T \simeq \frac{1}{2}m_0v^2. \tag{3}$$

Let us first show that at $\beta = 0.3$, the error in using the latter expression instead of the former one is pretty small. Let us write the exact expression of Equation 2 in terms of β.

$$T = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)m_0c^2. \tag{4}$$

Where $\beta < 0.3$, which is always true in the non–relativistic regime, we can expand $1/(\sqrt{1-\beta^2})$ using the binomial theorem,

$$\begin{aligned} \frac{1}{\sqrt{1-\beta^2}} &= (1-\beta^2)^{-\frac{1}{2}} \\ &= 1 + \left(-\tfrac{1}{2}\right)(-\beta^2) + \tfrac{1}{2!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)(-\beta^2)^2 + \cdots . \\ &\simeq 1 + \tfrac{\beta^2}{2} + \tfrac{3\beta^4}{8}. \end{aligned} \tag{5}$$

Therefore, we can write

$$\gamma - 1 = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \simeq \frac{\beta^2}{2} + \frac{3\beta^4}{8}. \tag{6}$$

Putting the m_0c^2 back again, we get

$$T = (\gamma - 1)m_0c^2 \simeq \frac{1}{2}(m_0c^2)\beta^2 + \frac{3}{8}m_0c^2\beta^4. \tag{7}$$

Putting in $\beta = v/c$ we get

$$\begin{aligned} T &\simeq \tfrac{1}{2}m_0c^2\left(\tfrac{v^2}{c^2}\right) + \tfrac{3}{8}m_0c^2\left(\tfrac{v^2}{c^2}\right)\beta^2. \\ &\simeq \tfrac{1}{2}m_0v^2 + \tfrac{3}{8}m_0v^2\beta^2. \end{aligned} \tag{8}$$

The second term on the right represents a the leading correction term compared to just using $1/2m_0v^2$ for the kinetic energy. There are additional errors contributed by higher order terms in the binomial expansion, but these errors are order β^4 and above, and are therefore much smaller than the leading term for $\beta <$

0.3. The fractional error σ_T/T by just assuming $T = 1/2m_0v^2$ is the ratio of the leading order error to $\frac{1}{2}m_0v^2$, or

$$\frac{\sigma_T}{T} = \frac{\frac{3}{8}m_0v^2\beta^2}{\frac{1}{2}m_0v^2} = \frac{3\beta^2}{4}. \tag{9}$$

At $\beta = 0.3$, the largest velocity still in the non-relativistic regime, σ_T/T is just under 7%. This is not an insignificant error, but it's acceptable to me for this course. If β is larger than 0.3, σ_T/T starts to become too big to be acceptable, and you must use a more precise, relativistic expression for T.

One nice outcome of this analysis is that you can see from it how to get the kinetic energy of a particle whose rest mass and β you know, without ever converting back to silly SI units. Consider Equation 8, and take only the leading order term. This is $1/2m_0v^2$ but written in terms of quantities we know in eV units. So, suppose we have a proton of mass 938 MeV/c^2 moving at 0.3 of the speed of light. Then $E_R = \mathrm{m_0c^2} = 938\,\mathrm{MeV}$ and $\beta = 0.3$ so

$$T \simeq \frac{1}{2}m_0v^2 = \frac{1}{2}(m_0c^2)\beta^2 = \frac{1}{2}E_R\beta^2 = 0.5\times 938[\mathrm{MeV}]\times 0.3^2 = 282\,\mathrm{MeV}. \tag{10}$$

The error in this estimate of the kinetic energy is about 7%. Use of equation 2 would yield the exact kinetic energy; use of the approximate expression of Equation 8 would yield an error much smaller than 7%. Notice that the kinetic energy is significantly smaller than $E_R = 938\,\mathrm{MeV}$, which is what you expect for a particle in the non-relativistic regime.

Another nice thing about this method is that the only dimensionful quantities in the formula are energies, so in fact the energy can be in any units you like. For example, if an object has a energy at rest of 0.9 horsepower-hours (an old unit of energy - 1 joule is 3.8×10^{-7} horsepower-hours), and its velocity is 0.1c, then its kinetic energy is $0.5 \times 0.9[\mathrm{horsepower} - \mathrm{hours}] \times 0.1^2$, which is 0.0045 horsepower-hours. No unit conversions were required!

In both the mildly relativistic and extremely relativistic regimes, the kinetic energy is compatible with or larger than the rest energy. In these regimes, the kinetic energy must be calculated using Equation 2. So, suppose $\beta = 0.5$, and we have a particle of rest mass 135 MeV/c^2. What is its total energy and what is its kinetic energy? First, $\beta = 0.5$ puts us in the mildly relativistic regime. So to calculate γ we use $\gamma = 1/\sqrt{(1-0.5^2)} = 1/\sqrt{(1-0.25)} = 1/\sqrt{(3/4)} = 2/\sqrt{3} = 1.15$. Therefore $E = 1.15 m_0 c^2 = 1.15 \times 135\,\mathrm{MeV} = 155\,\mathrm{MeV}$. The kinetic energy is $T = (\gamma - 1) m_0 c^2 = \gamma m_0 c^2 - m_0 c^2 = E - m_0 c^2 = (155 - 135)\,\mathrm{MeV} = 20\,\mathrm{MeV}$. So in this case the kinetic energy is 15% of the total energy.

Finally, let's do a highly relativistic one. Suppose we have a proton having $\gamma = 90$. What is its total and kinetic energy? Total energy is just γ times rest energy, or $90 \times 938\mathrm{MeV} = 84.4\,\mathrm{GeV}$. To get the kinetic energy, just subtract the rest energy (938 MeV) from the total energy, but this makes a very small difference, just over 1%. For a highly relativistic particle, the kinetic energy is almost exactly equal to the total energy, with the discrepancy equal to the rest energy. In other words, highly relativistic particles behave almost exactly like photons. The energy stored in their rest mass is almost negligible. This is one reason why theoretical physicists hope to unify physics at high energies; it doesn't matter 'up there' that all the particles have different rest masses! Those parameters simply aren't important for dynamics at sufficiently high energy.

2 Relativistic Dynamics

In relativity it is very rare indeed to have any static potentials, such as a gravitational potential or an electrostatic potential to worry about. The reason is that potentials are not actually consistent with Einstein's postulates. To see why this is, recall that the gravitational force between two bodies has magnitude GM_1M_2/r^2 where G is Newton's gravitational constant, M_1 and M_2 are the masses of the bodies and r is their separation. Now imagine that r is, say 30 kpc, roughly the dimension of the luminous component of our galaxy. Now let one of the bodies disappear suddenly. If Newton's law of gravitation is correct, the force on the other mass disappears immediately! This is inconsistent with Einstein's basic idea that nothing can travel faster than c, since if Newton's law of gravitation is right, then the gravitational field can transmit information from place to place instantaneously. So, in relativity, static fields are replaced by propagating particles which carry the forces from place to place, never having a measured velocity faster than c. At least, this is how it looks from an experimental perspective. Some of you may have read recently about an experiment called OPERA in which neutrinos generated at CERN were fired through the Earth to a detector in Gran Sasso (an underground lab), and there were reports that these neutrinos appear to travel faster than light! This result is highly controversial, and until it is verified by another group, we will assume that it will turn out to be a mistake. If it IS verified by others, the whole theory of relativity will be in need of modification to account for the new affect. That's science.

Because there are no potentials, there are no conventional forces either. Instead, bodies propagate from place to place in straight lines until they undergo interactions, which we may consider to be approximately point scatters. Or, sometimes, a single body may break up or decay into two, or perhaps many, bodies. A great proportion of problems in special relativity has to do with studying the dynamics of collisions and decays. If you know the initial state of a system - say you have some particle at rest with respect to an observer O, and then something happens, say the particle decays into two bodies, what can you say about the final state of the system, from the perspective of observer O?

To solve problems like this in pre-relativistic physics, one of the key tools we have is conservation of energy. In classical mechanics, energy that be converted into forms that we can't measure. For example, two sticky balls can collide and adhere to each other. In this collision, the total kinetic energy of the balls after the collision may be less than the sum of their kinetic energies before the collision. That's because after the collision, the process of sticking them together heats the balls up, so some of the kinetic energy before the collision is converted into heat after. We say the collision was inelastic.

In relativistic dynamics, we still talk about elastic and inelastic collisions. An elastic collision is where you have the same particles before the collision as you have after it. Two electrons scattering off each other is an example of an elastic collision. Two electrons before, two electrons afterwards. Inelastic collisions are processes where the particle species before the collision and after the collision are different. For example, if you collide a high energy electron with a high energy positron, you may create a muon and an antimuon in final state. This is an example of inelastic scattering. Another example of inelastic scattering a bit more like the sticky ball problem is neutron capture by boron. A nucleus of ^{10}B can absorb a neutron and break up to yield a nucleus of ^{7}Li, an α–particle, and a γ–ray photon. Again, the particles in the initial and final states are different, so this is inelastic scattering. This process is very important in nuclear reactors, where control rods made of ^{10}B are lowered into the reactor core to absorb some of the neutrons liberated in nuclear fission, thereby reducing the reaction rate and cooling the core.

One final note about energy. What about massless particles? They are special, because they travel at the speed of light, therefore to any observer they have infinite γ. However, this isn't a problem. The reason is, we know they are massless. There is no point in breaking their total energy into two components, kinetic and mass energy. They don't have any mass energy, because $m_\gamma = 0$. Therefore, for a photon $T = E$. The formula for kinetic energy in terms of m_0c^2 used for massive particles is useless here because $m_0 = 0$. If this is puzzling, think of a massless particle as a limit of a very light particle. For a given energy, as particles with that energy get lighter and lighter, their rest energies m_0c^2 get smaller and smaller, and their gamma factors get bigger and bigger, such that $\gamma m_0 c^2$ is fixed. So you can have $\gamma \to \infty$ at the same time as $m_0 \to 0$, and the product of these two quantities can remain finite.

3 Energy Conservation in Special Relativity

Enough discussion! The key question is, can we use energy conservation to solve problems in special relativity? Yes, but with one important proviso. Energy is only conserved if it is the same observer O measuring the energy of all the particles before the dynamic event (collision or decay), and measuring the energies of all the final state particles after the event. The sum of all the energies before the event is the same as the sum of all the energies after the event, as long as the same observer O measures the energies in both cases.

So if we have, for example, a π^0–meson at rest with respect to some observer O (the mass of a π^0–meson is $135\,\mathrm{MeV/c^2}$), and this meson decays into two γ–rays (photons), what is the total energy of the two γ–rays? Clearly the energy of the π^0–meson initially in the rest frame of O is $135\,\mathrm{MeV}$, therefore the sum of the energies of the two decay photons is also $135\,\mathrm{MeV}$.

Intuitively you would also guess that each of the photons wind up with the same energy, so 77.5 MeV each, but there's no way of proving that without also knowing that momentum os also a conserved quantity. We haven't discussed momentum yet, so let's go on and do that now.

4 Momentum

The momentum of a particle in relativity is

$$\vec{p} = \gamma m_0 \vec{v}. \tag{11}$$

The high energy physics units for momentum are eV/c. This means that $\vec{p}c$ has eV units. From Equation 11 we obtain an expression for $\vec{p}c$,

$$\vec{p}c = \gamma m_0 \vec{v} c = \gamma \, m_0 c^2 \left(\frac{\vec{v}}{c} \right) = \beta \gamma m_0 c^2 \hat{v}, \tag{12}$$

where $\hat{v}$ is a unit vector in the direction of motion of the particle. I emphasise, because sometimes it will matter, that as in pre-relativistic classical mechanics, momentum is a vector quantity, having three components which are often expressed in Cartesian coordinates.

Let's study momentum a little. Consider a proton having a total energy of 941 MeV. What is its momentum? Firstly, recall that the proton rest mass is 938 MeV, so that the kinetic energy is only 3 MeV. Therefore we are in the almost completely non-relativistic regime, and we may set $\gamma = 1$. In this regime, the magnitude of the momentum is

$$pc = \beta \gamma m_0 c^2 \simeq \beta m_0 c^2 = \beta E_R, \tag{13}$$

so we need an expression for β. To get this, note that we are in the low energy limit, so that the kinetic energy is approximately

$$T = \frac{1}{2} m_0 c^2 \beta^2 = \frac{1}{2} E_R \beta^2 \tag{14}$$

so that

$$2TE_R = (E_R \beta)^2 \tag{15}$$

and therefore

$$pc \simeq \beta E_R = \sqrt{2TE_R} = \sqrt{2 \times 3[\text{MeV}] \times 938[\text{MeV}]} = 75\text{MeV}. \tag{16}$$

Therefore the momentum is $p = 75\,\text{MeV/c}$. That's it. Again, no need to convert back to SI units, and the calculation is a lot simpler, and your answer almost certainly more accurate, if you don't.

Next, let's do a highly relativistic example. Suppose we have a proton with a total energy of 90 GeV. What is it's momentum? First, find γ by noting that $E = \gamma E_R$, therefore $\gamma = E/E_R = 90,000/938 = 95.9$. This is much greater than 7, so we are indeed in the highly relativistic regime, and $\beta = 1$. The momentum-energy is

$$pc = \beta\gamma m_0 c^2 \simeq \gamma m_0 c^2 = E_R \tag{17}$$

So, in the high energy limit $pc = 90\,\text{GeV}$, or $p = 90\,\text{GeV/c}$. In the high energy approximation, once again, particles behave as if they were massless! They are photon-like. You may remember that for a photon, the momentum and energy are related by $E = cp$. This is an exact formula for a massless particle; it's also approximately true for any particle in the highly relativistic regime.

Finally, time to do a mildly relativistic example. Suppose I have a proton again, this time with a momentum of $p = 0.8\,\text{GeV/c}$. What is β? Since pc is of the same order as E_R at 938 MeV, I'm suspecting this is in the mildly relativistic regime, so I need the full formula for momentum,

$$pc = \beta\gamma m_0 c^2 \tag{18}$$

Therefore $\beta\gamma = (pc/E_R) = 2000/938 = 2.13$. We therefore have to solve

$$2.13 = \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}. \tag{19}$$

You can't set either β or γ to 1, you just have to solve it. Square both sides and rearrange to get $(2.13)^2 - (2.13)^2\beta^2 = \beta^2$, or $(2.13)^2 = (1 + (2.13)^2)\beta^2$, or $\beta = 2.13/\sqrt{1 + (2.13)^2} = 0.9c$. So we are indeed in the mildly relativistic regime.

5 Momentum conservation

Like energy, total momentum as measured by the same observer O before and after an interaction is the same. Unlike energy, however, momentum is a vector. The sum of the momentum vectors of all the particles before and after the collision will be the same, as long as all those momentum vectors are determined by the same observer, non-accelerating throughout.

Let us see how conservation of momentum helps us with our π^0–meson decay. Consider an observer O at rest with respect to the π^0–meson before it decays. The momentum is zero. Therefore, the sum of the momenta of the two photons after the decay must be zero. If they have equal and opposite momenta, then their energies are equal, because $E = c|\vec{p}|$ for a photon, and the momenta differ only by a sign. Therefore, the two decay photons have equal energies, 77.5 MeV each, and are emitted back-to-back in the rest frame of observer O.

What about some other observer, moving with respect to the π^0–meson? Well, to these observers the energy before the collision is greater than the rest energy of the π^0–meson, because it's moving. And, the momentum of the π^0–meson is not zero, again because it's moving. So, both the initial conditions and the outcome look different to an observer with respect to whom the initial π^0–meson is moving. More about this next time.