Lecture 1 - Introduction, then Beta (β) and Gamma (γ)

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1 Overview of this course

This year is the first that I am starting to teach special relativity in year 2 having already taught it to the same group during year 1. The notes that I gave out last year for PHY101 will be useful to you. They are the same notes that are still on the PHY101 web site (under MOLE) this year. Some slightly more advanced notes, but with a very large overlap with last years PHY101 course, are also available on the PHY206 web site. These also have not changed since last year.

However, it would be silly to teach the same course again, since it has such a large overlap with the PHY101 lectures you have already had. So what I intend to do instead is to re-teach some of the techniques necessary to do the types of problems most relevant to further work with special relativity, and to introduce a few new concepts. The emphasis will be on the former. Experience with examining PHY101 and PHY206 over the past two years has taught me that students tend to ignore my advice on how best to get the right answers to calculations, preferring instead a set of 'A level' style methods which I know don't work because I observer their success rate on your exam scripts! Here are some of these 'dodgy' and 'dubious' methods.

- **Heinous crime against physics number 1** The use of inappropriate non-relativistic formulae for relativistic problems. Assuming that kinetic energy is $T = 1/2\,mv^2$

or momentum is $p = mv$. These expressions are approximations which are only useful for particles moving much slower than light. You've come to rely on them because most of the particles you have met are slow moving, but don't think they are universal and be particularly careful to justify their use whenever you invoke them in this course.

- **Heinous crime against physics number 2** Using calculators to evaluate β given a value of γ and vice versa, when the calculation is either extremely relativistic or almost entirely non relativistic. This often fails because most calculators have trouble evaluating things like $1 - 10^{-12}$, and you yourselves have trouble entering expressions like `1-0.9999999823` into your calculator! Expressions like this come up often if you don't know how to avoid them.

- **Heinous crime against physics number 3** Failure to make appropriate approximations, particularly failure to recognise when one can set β or γ equal to 1 and considerably simplify an expression.

- **Heinous crime against physics number 4** Conversion of energies in eV, MeV, or GeV, into joules (J), masses in $\mathrm{eV/c^2}$, $\mathrm{MeV/c^2}$, $\mathrm{GeV/c^2}$ into kg, and momenta in eV/c, MeV/c and GeV/c into $\mathrm{kg\,m\,s^{-1}}$, performance of relativistic calculations in inappropriate SI units, then attempts to convert the result back into electron volts. In over 50% of cases, this results in the wrong answer, usually by one or several factors of c, so the answer will be off by about 10^8, 10^{16}, etc. And, the calculations in SI units are far longer and therefore waste your precious time.

- **Heinous crime against physics number 5** Confusion between the different energies in physics, rest energy, kinetic energy, and total energy. Failure to appreciate the differences between results that are general for all particles, those that apply only to non-relativistic particles (like the kinetic energy of a particle that has $v \ll c$), and those that apply only to very highly relativistic particles like photons, such as energy $E \simeq cp$, where p is momentum.

- **Heinous crime against physics number 6** Assuming that all you need to know to do relativity problems is Lorentz transformations of x and ct, Lorentz contraction, and time dilation. Most relativity calculations are concerned with energy and momentum, not position and time. So if this is all you know, you will have trouble with my problems.

In spite of my having gone on at great length in favour of you all abandoning your lives of crime, many of you persisted in the exam in some or all of the above destructive practices. One aim of this course is to do enough problems, and to encourage you also to do enough problems, to make all of you (hopefully) aware of what I consider (based on my experience) good practice, and frankly to get you all out of some of these bad habits.

Secondly, I will introduce some new material, mostly in the area of four-vectors and Lorentz invariant quantities, that are of interest to practitioners of special relativity, particularly those of you who will end up working at particle beam accelerators such as ATLAS. Hopefully this will be fun for people, and these are powerful techniques which are standard tools in the field.

1.1 Assessment

As you know, this course is half of a 10 credit module, sharing its module code with atomic physics. The overall assessment of PHY206 is 70% exam-based, with the remaining 30% split between 10% on each of two homeworks and 5% on each of two problems classes. Stathes and I split this exactly 50/50, so that half the exam is on relativity and half on atomic physics, we set one homework each, and we usually split each of the two problems classes into 50% atomic physics and 50% relativity.

Be warned that, given the nature of the taught material, I will consider it entirely legitimate to give students zero who get the wrong numerical answer because they tried to do the calculation in SI units and did the conversions incorrectly. I do this not out of malice, but because I genuinely consider that learning to handle the new units is one of the main learning outcomes of the course. I will give partial credit only where I consider that the student has tried to perform the calculation using the methods I am trying to teach you. I will of course give full credit to those of you who get the correct answer, using any correct working in any units, but those who choose to ignore my advice and persist with SI units will, in my experience, get zero credit in more than 50% of cases, particularly on the exam. I will also award no credit to a correct answer with no supporting calculations, or calculations that are inconsistent with the answer.

2 Beta (β) and gamma (γ)

The critical questions to ask when figuring out whether you need to tools of relativity are, firstly, what are the speeds v of the particles concerned relative to c? And, secondly, even if the particles are all moving with $v \ll c$, are the units employed in the question such that we may still benefit from a special relativistic approach to calculations? Because we are comparing v to c so much, we define

$$\beta = \frac{v}{c}. \tag{1}$$

The velocity v is therefore related to β by $v = \beta c$. If v approaches c then β approaches 1. β is exactly 1 only for massless particles, though as we shall see there are a large class of problems where massive particles have so much energy that they have β that is almost exactly 1, and in these situations it is often an enormous simplfication to set $\beta \sim 1$ for the purposes of doing calculations. The other quantity that comes up a lot, related to v or β is γ, which is defined as

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1+\beta)(1-\beta)}}. \tag{2}$$

The last form of γ is often useful when you are trying to simplify expressions with lots of βs and γs in them.

Because v is always between $-c$ and c, γ is always between $+1$ (for $v = 0$) and $+\infty$ (as $v \to c$). In fact, we can easily plot γ as a function of β, and this is what I have done below in Figure 1.

There are a few things that are worth noticing on this plot. Firstly, when $\beta < 0.3$, γ is 1 to within 5%. Secondly, when $\gamma > 7$, β is 1 to within 1%. I wish you to memorize these limits.

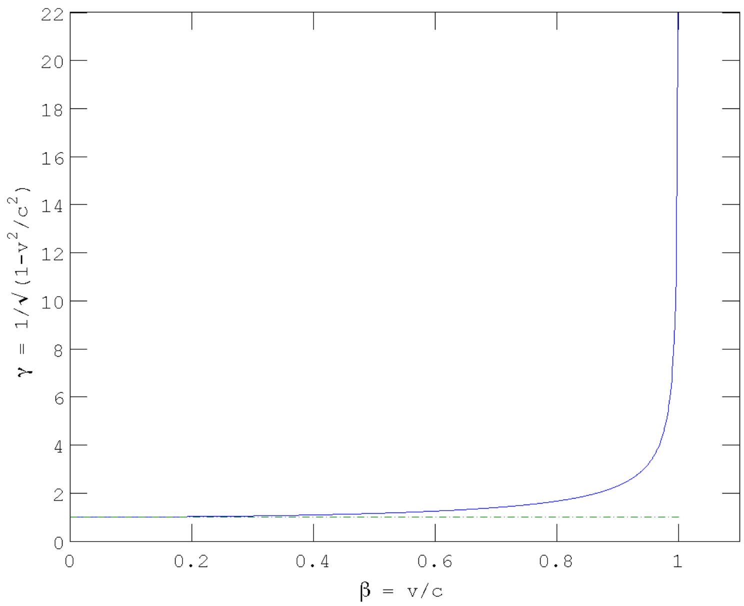

Figure 1: A plot of γ vs. β.

2.1 The non-relativistic regime

With $\beta < 0.3$, the effects of special relativity, which are only noticable when γ is significantly different than 1, are at the less than 5% correction level. We will call this the non-relativistic regime. This does not necessarily mean that it is best to convert all quantities back into SI units and start using your old A level 'skills'. There is a large class of problems, particulary α–decay of nuclei, in which relativistic-style calculations using energies in eV, etc, will get you to the right answer faster and with less possibility for mistakes than plugging in to your calculator and converting the mass of the alpha particle into kilograms, for example. We'll do plenty of examples of calculations with slow moving alpha particles emitted by decaying nuclei where the easiest way to figure out the dynamics is to approach the problem using relativistic units and notation.

2.2 The mildly-relativisitic regime

This is a term I invented myself; don't expect to see it in the literature. This regime is characterised by $\beta > 0.3$ and less than

0.99. In other words, the two boundaries of this regime are at $(\beta, \gamma) = (0.3, 1.05)$ and $(\beta, \gamma) = (0.99, 7)$. For calculations, this is the most difficult range of velocities, because you can neither set $\beta \sim 1$ or set $\gamma \sim 1$. However, in this regime, you can freely use the formula of Equation 2 to deduce γ from β and vice versa.

2.3 The ultra-relativistic regime

The final regime is characterised by particles having $\beta > 0.99$ and $\gamma > 7$. These particles are ultra-relativistic because the large size of γ means that relativistic effects dominate. Also, the velocity of such particles is approaching that of light. For ultra relativistic particles, I will rarely ask you to compute the particles velocity. I'm not really interested in answers like $2.999453213 \times 10^8\,\mathrm{m\,s^{-1}}$. I'm no better than you at counting the nines, and I have a horror of using so many significant figures. How often do you see c quoted to this number of figures? It's usually meaningless to do so. Instead, I might often ask you by how much is a particles velocity less than that of light. If you remember, in PHY101, we worked out and exploited an expression for getting the deviation from c. Let $\varepsilon = c - v$, and suppose we are in the ultra-relativistic regime, where $\epsilon \ll c$. Let us write down γ in terms of ε, as we did last year, and recall what happens.

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{1}{\sqrt{1-\frac{(c-\varepsilon)^2}{c^2}}} \\
&= \frac{1}{\sqrt{\frac{c^2-(c-\varepsilon)^2}{c^2}}} \\
&= \frac{c}{\sqrt{c^2-(c^2-2c\varepsilon-\varepsilon^2)}} \qquad (3) \\
&\simeq \frac{c}{\sqrt{2c\varepsilon}} \\
\gamma &\simeq \sqrt{\frac{c}{2\varepsilon}} \\
\gamma &\simeq \sqrt{\frac{c}{2(c-v)}},
\end{aligned}
$$

or

$$(c - v) = \varepsilon \simeq \frac{c}{2\gamma^2}. \tag{4}$$

Because $\varepsilon \ll c$ I have been able to neglect ε^2 when it appears on the fourth line of the algebra in Equation 3. These expressions relating γ and the difference between the particle speed and that of light, $c - v$, are only useful or correct in the ultrarelativistic regime. Do not try and use them for particles having $\gamma < 7$ or $\beta < 0.99$. They will give incorrect results.

Figure 2 is a cartoon drawing on which I have overlaid a rough guide to the different regimes on the previous graph of γ vs. β. I hope it will be useful in helping you decide which approximations it is OK to make.

3 Examples of conversion between β and γ

A particle is moving at $v = 0.6c$ with respect to some observer O. What regime are we in, what are β and γ of the rest frame of this particle with respect to the observer O?

Solution. β is v/c, or 0.6. Referring to our cartoon drawing, we are in the mildly relativistic regime. And in this regime we just plug in to $\gamma = 1/\sqrt{1-\beta^2}$ to obtain a β of $1/\sqrt{1-0.36} = 1/\sqrt{0.64} = 1/0.8 = 5/4$. We check that γ is consistent with the mildly relativistic regime, which it is, because it's greater than 1.05 and less than 7.

A particle is moving at $v = 0.002c$ with respect to some observer O. What regime are we in, what are β and γ of the rest frame of this particle with respect to observer O?

Solution. β is 0.002. This is the non relativistic regime. To all intents and purposes, $\gamma = 1$. You could figure out exactly what it is using a binomial expansion, and the answer you would get would be 1 plus a tiny quantity, of no real interest. Just say $\gamma = 1$ and go on to the next bit.

A particle is moving with a γ factor of 10 with respect to some observer O. What is β? How much slower is it moving than c,

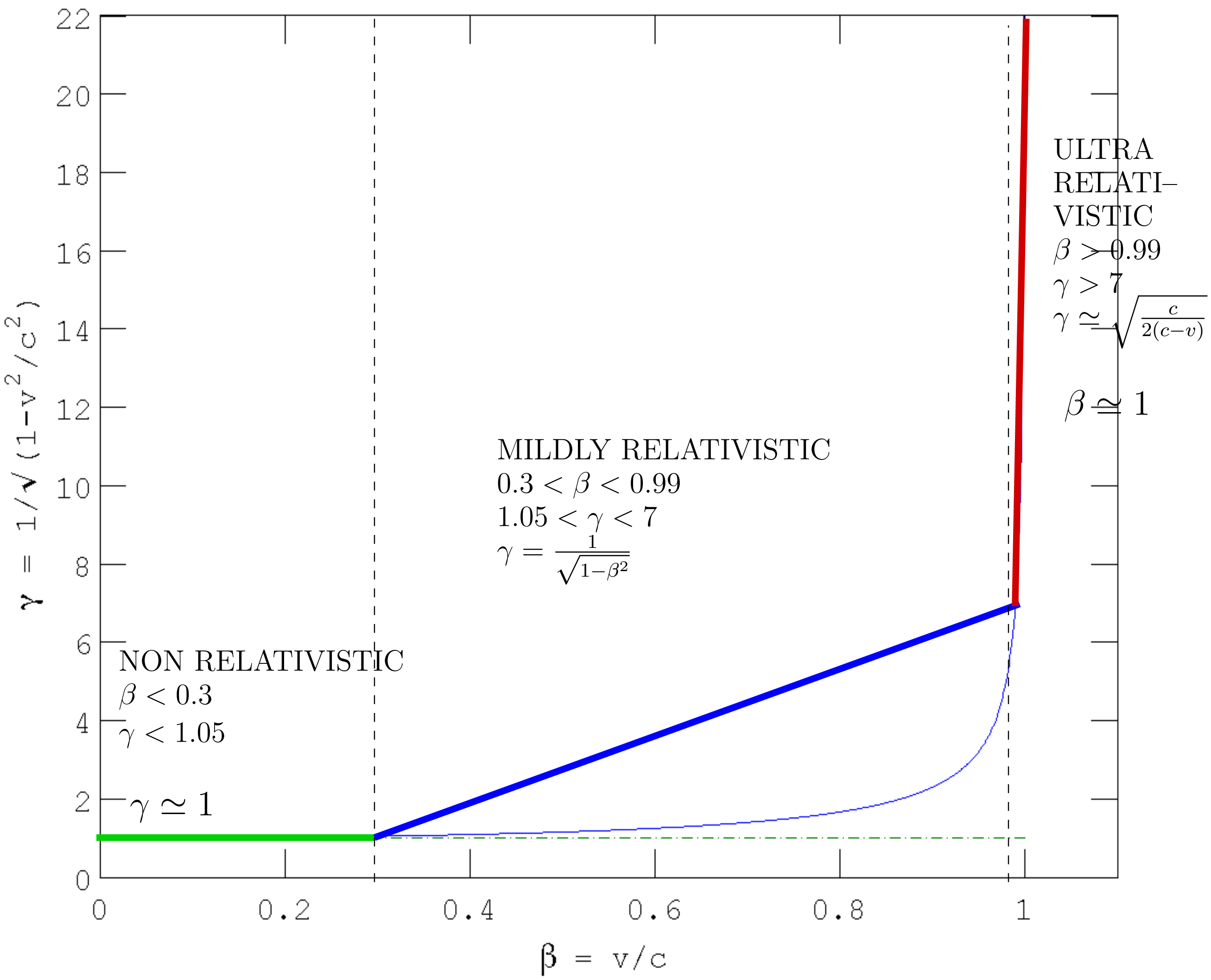

Figure 2: A cartoon drawing of the ranges of β and γ associated with the non-relativistic, mildly-relativistic, and ultra-relativistic regimes.

in $\mathrm{m\,s^{-1}}$*, with respect to observer O?*

Solution. We're in the highly relativistic regime because $\gamma > 7$. First, let's do this the hard way and see what a pain it is. If $\gamma = 10$, then $\sqrt{1-\beta^2} = 0.1$. Therefore $1 - v^2/c^2 = 0.01$, so that $v^2/c^2 = 0.99$, so that $v/c = 0.99498$. This means that $c - v = c(1-\beta) = c(1 - 0.99498) = 1.5 \times 10^6\,\mathrm{m\,s^{-1}}$. Now let's do this the easy way. Using Equation 4, we can write $c - v \simeq c/2\gamma^2 = c/200 = 1.5 \times 10^6\,\mathrm{m\,s^{-1}}$. No nines to write out, and just as accurate of an answer. In fact, more accurate, and it's far less likely that you'll make a mistake.

A particle is moving with $\gamma = 10^{12}$ *with respect to some observer O. Repeat the above question.*

Solution. We're in the highly relativistic regime because $\gamma > 7$. Let's do this the hard way and see that it's basically impossible. If $\gamma = 10^{12}$ then $\sqrt{1-\beta^2} = 10^{-12}$, so that $1 - v^2/c^2 = 10^{-24}$, or $v^2/c^2 = 1 - 10^{-24}$. If you plug this into your calculator, you'll conclude, wrongly, that $v = c$. Instead of doing this, use Equation 4 again, to write $c - v = c/2\gamma^2 = 3 \times 10^8\,\mathrm{m\,s^{-1}}/2 \times 10^{24}$, or $c - v = 1.5 \times 10^{-16}\,\mathrm{m\,s^{-1}}$. This is the difference between the speed of this particle and the speed of light in the reference frame of observer O.

4 Examples of deducing β and γ from various quantities with the dimensions of energy

Frequently in relativity problems, you are given some combination of total energy, kinetic energy, and mass of a particle at the beginning. From any two of these numbers you can work out γ. Once you have γ, you can figure out what regime you are in and make appropriate approximations to simplify the work. However, if you get γ wrong, you are completely sunk. So, please, practice working out γ. Here are some examples.

A proton (rest mass $938\,\mathrm{MeV/c^2}$*) has a total energy of* $20\,\mathrm{GeV}$ *in the coordinate system of some observer O. What is* γ *for the coordinate system at rest with respect to the proton?*

Solution. In this problem, we need to recall the relativistic expression for the total energy E and rest energy E_R of a particle. They are:

$$E = \gamma m_0 c^2 \tag{5}$$

$$E_R = m_0 c^2 \tag{6}$$

If the rest mass is $m_0 = 938\,\mathrm{MeV/c^2}$, then the rest energy is $E_R = m_0 c^2 = 938\,\mathrm{MeV}$. That's it! There is NO NEED to actually multiply by 3×10^8, because multiplying by c is accomplished by crossing out the /c. The number stays the same, just the units change. I wish I had a quid for every student who forgets this and actually gets their calculator out to multiply by c when converting from rest mass in eV/c to rest energy in eV. NO! Don't do that! Just let the units take care of it.

Next, we combine the equations for rest mass and rest energy to obtain $E = \gamma E_R$. This means that γ is just the ratio of the total energy to the rest energy, or equivalently the total energy in eV divided by the rest mass in eV/c^2. So in this problem, $\gamma = 20,000/938 = 21.3$, so we are in the highly relativistic regime, and $\beta \simeq 1$. It's that easy!

A proton has a kinetic energy of 3 GeV in the coordinate system of some observer O. What is γ, and what regime are we in?

Solution. The relativistic formula for the kinetic energy T is

$$T = (\gamma - 1) m c^2. \tag{7}$$

The proof of this is easy. There are only two forms of energy that a particle can actually have in special relativity problems, mass-energy and kinetic energy. Therefore, they have to add to the total energy. This leads to $T + m_0 c^2 = \gamma m_0 c^2$, and rearranging we obtain Equation 7. That's it. Combining Equation 7 with Equation 6 we get

$$T = (\gamma - 1) E_R. \tag{8}$$

Therefore,

$$\gamma = 1 + \frac{T}{E_R} = 1 + \frac{T}{m_0 c^2}. \tag{9}$$

In this problem, $\gamma = 1 + 3/0.938 = 4.2$. That's it.

Finally, we might also be given the total energy and the kinetic energy, but not the rest energy. In this case, you could just subtract $E - T$ to obtain $m_0 c^2$, and substitute E and $E_R = m_0 c^2$ into $\gamma = E/E_R$. This is the same as writing $\gamma = E/(E - T)$. So we now have lots of ways of obtaining γ from various energies.

$$\gamma = \frac{E}{m_0 c^2} = 1 + \frac{T}{m_0 c^2} = \frac{E}{E - T}. \tag{10}$$

We'll stop here for now. Next time we'll go on to discuss the relationship between energy, momentum and mass, and do more problems in that area.