Lecture 9 - Applications of 4–vectors, and some examples

E. Daw

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1 Review of invariants and 4–vectors

Last time we learned the formulae for the total energy and the momentum of a particle in terms of its rest mass m_0 and its velocity $\vec{v}$,

$$\begin{aligned} E &= \gamma m_0 c^2 \\ \vec{p} &= \gamma m_o \vec{v}. \end{aligned} \tag{1}$$

We also learned a key relationship between the total energy E, the momentum $\vec{p}$ and the rest mass m_0,

$$E^2 = |\vec{p}|^2 c^2 + m_0^2 c^4. \tag{2}$$

Finally we learned that we could construct 4–vectors out of the components of the spacetime coordinates of an event,

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z), \tag{3}$$

and out of the energy and momentum components,

$$(p^0, p^1, p^2, p^3) = (\frac{E}{c}, p^x, p^y, p^z). \tag{4}$$

We noticed that minus the square of the 0^{th} component, plus the sum of the squares of the rest of the components equals an invariant quantity. So, for example, $-c^2t^2 + x^2 + y^2 + z^2$ is an invariant, where (ct, x, y, z) are the coordinates of some event with respect to the origin. If $c^2t^2 > (x^2 + y^2 + z^2)$, then this quantity is equal to $-\tau^2$, where τ is the proper time interval between an event at the origin and the event having these coordinates. This means, physically, that there is some observer for whom both the events occur in the same place. For this observer, the spatial displacement between the events is $x = 0, y = 0, z = 0$, and hence the only displacement is a time displacement. But this is the very observer for whom the time displacement is the proper time interval, and we have just re-discovered that as long as the events are time-like separated, there is an observer that can get between the two events without exceeding the speed of light, who will measure the proper time interval, the minimum time interval between the events that any observer can measure. Furthermore, the existence of this observer implies that there can be a causal chain of events connecting the two events, so all observers therefore agree on which of the two events occurs first.

If $c^2t^2 < (x^2 + y^2 + z^2)$, there is no observer that can travel between the events since to do so he would need to exceed the speed of light. Events separated in this way are called space-like separated, and it is not true that all observers agree on which order space like separated events occurred.

A second invariant is formed from $-E^2/c^2 + (p^x)^2 + (p^y)^2 + (p^z)^2$, which is equal to $-m_0^2c^2$. In both cases, the invariant quantity can be written in matrix form as

$$\begin{pmatrix} u^0 & u^1 & u^2 & u^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}. \qquad (5)$$

The square matrix in the middle on the right is known as the Minkowski metric. Notice that as long as the minus sign is in the right position, the two 4–vectors separated by the metric need not be the same! So, for example, if you have a particle of energy E, momentum (p^x, p^y, p^z), and position (x, y, z) at time t, then $-Et + xp^x + yp^y + zp^z$ is an invariant quantity. You can prove this to yourself by Lorentz transforming the components of both position and momentum 4–vectors and verifying that the transformation matrices multiply out to give a result that depends neither on β nore on γ.

2 Applications of 4–momentum: conservation of the square of four-momentum

Now let us flex our new found muscles and see what can be done with 4–vectors. One of the key applications of 4–vectors is new conservation laws using invariants. Energy and momentum are conserved as long as the same observer makes the measurement before and after the process has taken place. The square of the 4 momentum (where by square I mean minus the square of the 0 component plus the sum of the squares of the other components) is the same in one coordinate system before a process as it is in a *different* non-accelerating frame of reference after the process. This is invaluable, especially in problems where one is dealing with the decay of a particle. For example:

Example 1 — A particle of rest mass M_1 decays into a particle of rest mass m_2 and a photon. In the rest frame of the particle produced in the decay, what is the energy E_γ of the photon, in terms of the rest masses M_1 and m_2 and the speed of light c?

We could solve this problem using straight energy and momentum conservation, but it's easier to use conservation of the square of the 4–momentum. Consider the decaying particle in its own rest frame. The total momentum is zero (it's not moving), and the total energy is M_1c^2. Therefore the initial total 4–momentum is $(E/c, p^x) = (M_1c, 0)$, and the square of this 4–momentum is $-M_1^2c^2$. Now, in the rest frame of the produced particle, it's energy is m_2c^2, since it is at rest, and the photon has energy E_γ and momentum E_γ/c. Then the 4–momentum is $(m_2c + E_\gamma/c, E_\gamma/c)$. Equating the squares of the 4–momentum of the decaying particle before the collision and the square of the sum of the 4–momenta of the decay particle and emitted photon after the collision, remembering that the square of the 0^{th} component gets a minus sign due to the sign of the 00 metric component, we get

$$
\begin{aligned}
-M_1^2c^2 &= -\left(m_2c+\frac{E_\gamma}{c}\right)^2+\left(\frac{E_\gamma}{c}\right)^2\\
&= -m_2^2c^2-2m_2E_\gamma-\left(\frac{E_\gamma}{c}\right)^2+\left(\frac{E_\gamma}{c}\right)^2\\
2m_2E_\gamma &= M_1^2c^2-m_2^2c^2\\
E_\gamma &= \frac{M_1^2c^2-m_2^2c^2}{2m_2}.
\end{aligned}
\tag{6}
$$

Notice that even though the initial conditions and the final conditions are specified in different frames of reference, there has been no need for a fiddly Lorentz transformation to get from one to the other. Instead, we exploited the fact that the square (in the usual sense) of the 4–momentum is the same to all observers, and conserved throughout the interaction.

Example 2 — A positron of total energy E strikes an electron that is stationary in the lab. The two particles annihilate and produce two photons, which in this particular instance (which is a special case), move off at equal angles θ on opposite sides of the axis along which the positron originally approached, as illustrated in Figure 1. Find an exact expression for the angle θ in terms of the energy E of the incident positron, the rest mass m_e of both the electron ad the positron, and c.

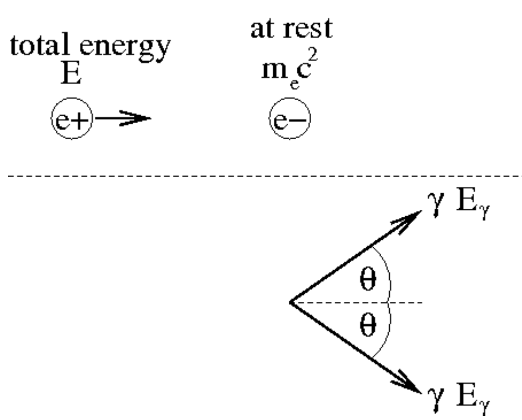

Figure 1: The collision of a moving positron (e^+) with an electron stationary in the laboratory. The two particles annihilate yielding two photons.

We first apply conservation of energy, which gives

$$E+m_ec^2=2E_\gamma. \tag{7}$$

The momentum of the incident positron can be written in terms of E and $m_e c^2$ using $E^2 = p^2c^2 + m_0^2c^4$, so that

$$pc = \sqrt{E^2 - m_e^2 c^4}. \tag{8}$$

The target electron is at rest so it has no momentum. To conserve momentum, the y–components of the momenta of the two photons must be equal and opposite. Furthermore, in this case, we are told that each of the emitted photons is emitted down a path that makes the same angle with the initial electron direction of incidence, as shown in the figure. This means that the y–components of the momenta must obey

$$p_{\gamma 1} \sin\theta = p_{\gamma 2} \sin\theta, \tag{9}$$

Therefore we must have

$$p_{\gamma 1} = p_{\gamma 2}, \tag{10}$$

and therefore the photon energies are also equal, so $E_{\gamma 1} = E_{\gamma 2} = E_\gamma$. So we again use conservation of momentum, this time along the beam axis. We equate the positron momentum of Equation 8 with the x–components of the momenta of the two photons to obtain

$$\sqrt{E^2 - m_e^2 c^4} = 2E_\gamma \cos\theta. \tag{11}$$

Dividing Equation 11 by Equation 7 eliminates the unknown E_γ and obtain

$$\cos\theta = \frac{\sqrt{E^2 - m_e^2 c^4}}{E + m_e c^2}. \tag{12}$$

Example 3 — An electron of total energy $E = 10\,\mathrm{GeV}$ *strikes a proton at rest in the lab. Using appropriate approximations to simplify the calculation where possible, what is the maximum total rest mass that can be created in an inelastic collision where both the electron and the proton are destroyed? The rest mass of the proton is* $m_p = (938\,\mathrm{MeV})/\mathrm{c}^2$.

The easiest way to do this problem is using 4–vectors. In this particular problem, the initial conditions are given in the lab frame, so let's write the total four momentum of the incident electron plus the proton target in the lab frame. We start by noticing that the incident electrons total energy is about 5000 times its rest energy, so we neglect the rest energy of the incident electron. Therefore the momentum of the incident electron is taken as E/c. The 4–momentum of the electron plus stationary target proton is therefore

$$\mathbf{p_i} = \begin{pmatrix} E/c + m_p c \\ E/c \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

Now, the maximum rest mass that can be created in the collision is determined by the total energy in the frame in which the final state particles are at rest. This is because any kinetic energy the final state has could potentially have been used to generate extra mass. The final state particles are only all at rest in the centre of mass frame. In this frame, the rest energy is Mc^2, where M is the sum of the masses of the particles that can be created, and the momentum is zero. Therefore we write out the 4–momentum of the final state in the centre of mass frame.

$$\mathbf{p_f} = \begin{pmatrix} Mc \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14}$$

Now, we calculate the combination of the 4–momentum components that is conserved in a frame independent manner. It is most important to realize that this only works because this combination of E and p is independent of the velocity of the inertial frame. It is not true that total energy and momentum before a collision in one frame are equal to total energy and momentum after the collision in a different frame. Energy and momentum conservation only apply where it is the same observer measuring energy and momentum before and after the interaction.

$$-\left(\frac{E}{c} + m_p c\right)^2 + \frac{E^2}{c^2} = -M^2 c^2. \tag{15}$$

Notice how simple this procedure is, because in the centre of mass frame at threshold where nothing is moving, the 4–momentum only has one non-zero component. Now rearrange and simplify.

$$\begin{aligned} -M^2c^2 &= -\tfrac{E^2}{c^2} - 2Em_p - m_p^2c^2 + \tfrac{E^2}{c^2} \\ Mc^2 &= \sqrt{2Em_pc^2 + (m_pc^2)^2} \\ &= m_pc^2\sqrt{\tfrac{2E}{m_pc^2} + 1}. \end{aligned} \tag{16}$$

Using $m_p c^2 = 938\,\mathrm{MeV}$ we obtain $Mc^2 = 4.43\,\mathrm{GeV}$.

3 The Greisen-Zatsepin-Kuzmin (GZK) limit

Example 4 — Cosmic rays consist of high energy protons emitted by astrophysical sources far from Earth. However, there is a theoretical limit to the maximum energy of cosmic rays from deep space caused by the cosmic microwave background radiation. If a proton has sufficient energy, it can scatter off a microwave background photon to form, for example, a neutron and a $\pi-$meson.

$$p + \gamma \rightarrow \pi^+ + n. \tag{17}$$

Deduce the maximum proton energy that can travel through deep space taking cosmic microwave background photons have energies of $6 \times 10^{-4}\,\mathrm{eV}$, and using 140 MeV and 938 MeV for the $\pi-$meson and proton rest energies, respectively. Assume that the neutron and proton rest energies are the same.

You can again use 4–momentum to do this problem. First, write down the total 4 momentum of the gamma and the proton. Use $\mathbf{p}_\gamma$ for the 4–momentum of the microwave photon, and $\mathbf{p_p}$ for the 4–momentum of the proton. The total 4–momentum of the initial state is therefore $(\mathbf{p}_\gamma + \mathbf{p_p})$. Now use a dot to represent the operation of taking minus the square of the zeroth component and adding to it the sum of the squares of the other three components. With this notation the intial state 4 momentum dotted with itself is:

$$(\mathbf{p}_\gamma + \mathbf{p_p}) \cdot (\mathbf{p}_\gamma + \mathbf{p_p}) = \mathbf{p}_\gamma \cdot \mathbf{p}_\gamma + \mathbf{2p}_\gamma \cdot \mathbf{p_p} + \mathbf{p_p} \cdot \mathbf{p_p}. \tag{18}$$

Now deal with these three terms on the right in order. $\mathbf{p}_\gamma \cdot \mathbf{p}_\gamma = 0$, because it is equal to $-E_\gamma^2/c^2 + |\vec{p_\gamma}|^2$ which is zero because $E_\gamma = |\vec{p_\gamma}|c$. Similarly, $\mathbf{p_p} \cdot \mathbf{p_p} = -m_p^2 c^2$ because $-E_p^2/c^2 + |\vec{p_p}|^2 = -m_p^2 c^2$. Therefore

$$(\mathbf{p}_\gamma + \mathbf{p}_\mathbf{p}) \cdot (\mathbf{p}_\gamma + \mathbf{p}_\mathbf{p}) = 2\mathbf{p}_\mathbf{p} \cdot \mathbf{p}_\gamma - m_p^2 c^2. \tag{19}$$

What about the 4–dot product between the four momenta p_p and p_γ? Do everything in one dimension, since at threshold for the production of new particles, the collisions will be head on. Then the incoming proton has energy E_p and momentum $+E_p/c$, neglecting rest mass because the cosmic ray energy is enormous. Similarly, the photon has energy E_γ and momentum $-E_\gamma/c$. Forming the 4–dot product of these two we get

$$\mathbf{p}_\gamma \cdot \mathbf{p}_\mathbf{p} = -\frac{2E_\gamma E_p}{c^2}. \tag{20}$$

Therefore Equation 19 becomes

$$(\mathbf{p}_\gamma + \mathbf{p}_\mathbf{p}) \cdot (\mathbf{p}_\gamma + \mathbf{p}_\mathbf{p}) = -\frac{4E_\gamma E_p}{c^2} - m_p^2 c^2. \tag{21}$$

Now, as in example 2, at threshold the final state neutron and proton will be produced at rest in the centre of mass frame. Therefore the 4–momentum in the final state is

$$\mathbf{p}_\mathbf{f} = \begin{pmatrix} (m_n + m_\pi)c \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

Hence the 4–dot product of p_f with itself is $-(m_n + m_\pi)^2 c^2$. Equating this with the 4–dot product of the initial momentum with itself, we obtain

$$\begin{aligned} \tfrac{4E_\gamma E_p}{c^2} + m_p^2 c^2 &= (m_n + m_\pi)^2 c^2 \\ &= m_n^2 c^2 + 2m_n m_\pi c^2 + m_\pi^2 c^2 \end{aligned} \tag{23}$$

We assume $m_p^2 c^2 = m_n^2 c^2$ and rearrange to give

$$E_p = \frac{2(m_n c^2)(m_\pi c^2) + (m_\pi c^2)^2}{4E_\gamma} \tag{24}$$

Plugging in some numbers, I get $E_p = 1.2 \times 10^{20}\,\text{eV}$. A proton having this energy would have a gamma factor of 10^{11}. To an observer in the rest frame of this proton, our galaxy would appear Lorentz contracted from a diameter of about 30 kPc to a diameter of about $30,000[\text{pc}] \times 3.1 \times 10^{16}[\text{m/pc}]/10^{11} = 9.3 \times 10^9\text{m}$. Such a proton, in its own reference frame, would traverse our galaxy in 31 seconds.

Searches for ultra high energy cosmic ray protons have discovered events close to the GZK threshold; discovery of a significant population of protons having energies higher than this cutoff would indicate that the source of the cosmic rays is very close to us, something that we think is unlikely. Seeing whether cosmic ray protons do indeed cut off at about $10^{20}\,\text{eV}$ is an interesting current experimental problem.