Lecture 6 - The relativistic doppler shift of light

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1 Introduction

Today we will study the doppler effect, and in particular the redshift of light emitted by a source receding from an observer. The non-relativistic doppler shift may be familiar to you from your A–level studies, and indeed you may also have discussed Hubble's law, which concerns the redshift of light from galaxies receding from us due to the expansion of the Universe.

2 The non-relativistic doppler effect

We consider the case where a vehicle is receding from an inertial observer at a constant radial velocity $v = \beta c$ with $v \ll c$. Figure 1 shows a spacetime diagram for the recession of the moving source.

Let us consider the emission of two successive wavefronts in the wavetrain of the light from the moving source. The first of these wavefronts is emitted at the origin on the spacetime diagram. Since non relativistically all non-accelerating clocks run at the same rate, we may assume that the clock on the moving source determining when the next wavefront is emitted runs at the same rate as a stationary clock in the coordinate system, so the time of emission of the subsequent wavefront on either clock is τ_{em}. Therefore the ct coordinate of the emission of this wavefront is $c\tau_{\mathrm{em}}$. But

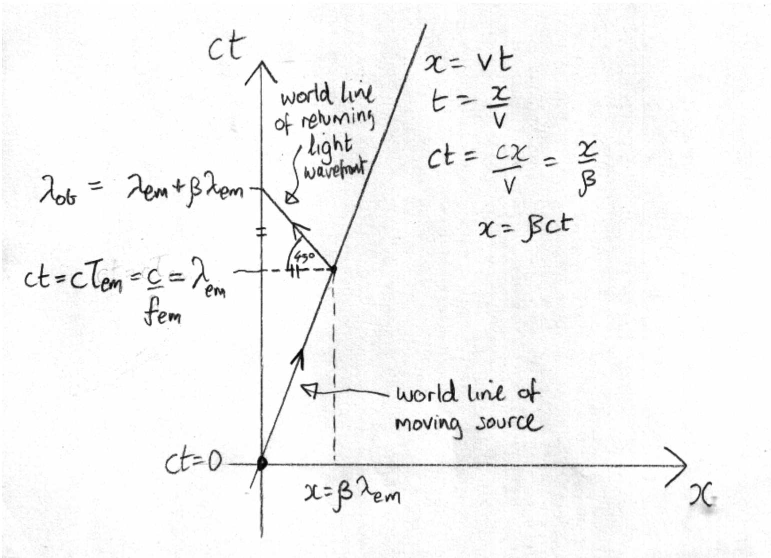

Figure 1: A source of constant velocity receding radially from an observer along the x axis at velocity $v = \beta c$. The events of the emission of two successive wave crests are marked with dots. The second of these wave crests is not coincident with the origin, so light from this wave crest travels back towards an observer at the origin along a world line at 45° to the horizontal. This spacetime diagram is drawn in the frame of reference stationary with respect to an observer at the origin at rest with respect to the detector measuring the frequency of the radiation from the receding source.

$$c\tau_{\mathrm{em}} = \frac{c}{f_{\mathrm{em}}} = \lambda_{\mathrm{em}}, \tag{1}$$

where f_{em} is the frequency of the source in its rest frame and λ_{em} is the wavelength of the source in its rest frame. The equation of the world line of the receding source is $x = vt$, or $ct = x/\beta$, where as last lecture $\beta = v/c$. Therefore we can rearrange and write $x = \beta ct$. So if $ct = \lambda_{\mathrm{em}}$, then $x = \beta\lambda_{\mathrm{em}}$. Next, notice that the triangle formed by the horizontal line at $ct = \lambda_{\mathrm{em}}$, the ct axis, and the world line of the returning light ray is right angled and isosceles, and therefore the ct coordinate of its point of intersection with the ct axis is $\lambda_{\mathrm{em}} + \beta\lambda_{\mathrm{em}}$. But if the time of emission of this wavefront times c was λ_{em}, then the time of observation is λ_{ob}, so that

$$\lambda_{\rm ob} = \lambda_{\rm em} + \beta\lambda_{\rm em}. \tag{2}$$

The redshift z is defined

$$z = \frac{\lambda_{\rm ob} - \lambda_{\rm em}}{\lambda_{\rm em}}. \tag{3}$$

Substituting Equation 2 into Equation 3 we obtain

$$z = \beta = \frac{v}{c}. \tag{4}$$

Note that the definition is such that if v is positive, the source is moving away from the observer, and the wavelength of the light gets longer. This means is that if the light starts out in the visible part of the electromagnetic spectrum, the Doppler effect moves its colour towards the red end. If v is negative, the source is moving twoards the observer and the light is blue shifted, reduced in wavelength, and increased in frequency.

This result is approximately correct for sources whose recession velocity v is much less than the speed of light. As usual, the rough rule of thumb is that if the $v < c/3$, the non relativistic result is sufficiently accurate for most purposes.

3 The longitudinal relativistic doppler effect

Once the source becomes relativistic, a more sophisticated analysis involving the Lorentz transformations must be used. This time, we start by considering the problem in the rest frame of the moving source. In this frame, the world line of the source follows the ct' axis, because it is not moving. As usual, we consider the origins of this primed coordinate system and the unprimed coordinate system of the observer from whom the clock is receding to be coincident at time $t = t' = 0$. Figure 2 shows the world line of the clock and the events of the emission of successive wave fronts in the primed rest frame of the moving source.

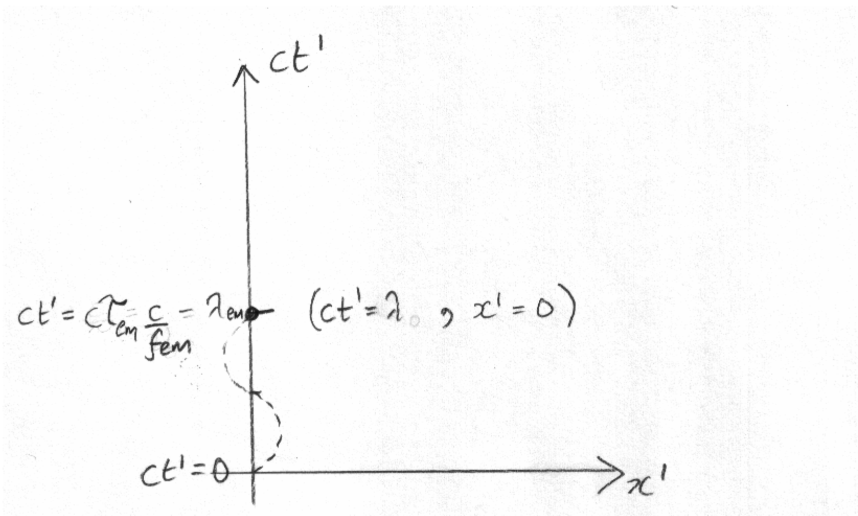

Figure 2: The emission of two successive wave fronts as events in the rest frame (primed) of the source.

As before, the first considered wave front is emitted at $ct' = 0$. In the rest frame of the source, the ct'–coordinate of the next wavefront is $ct' = c\tau_{\rm em} = \lambda_{\rm em}$. The x'–coordinate of both wavefronts is zero, since in this frame the source is at rest. Now let us recall our Lorentz transformations.

$$\begin{aligned} t' &= \gamma\left(t - \tfrac{vx}{c^2}\right), \\ x' &= \gamma\left(x - vt\right), \\ y' &= y, \\ z' &= z. \end{aligned} \tag{5}$$

Taking the first of these transformations, and inverting to express t in terms of t' and x', we get

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right). \tag{6}$$

For the second wavefront, noting that $x' = 0$, we get

$$ct = \gamma ct' = \gamma\lambda_{\rm em}. \tag{7}$$

This gives us the ct coordinate of the event of emission of the second wavefront in the unprimed coordinate system of the ob-

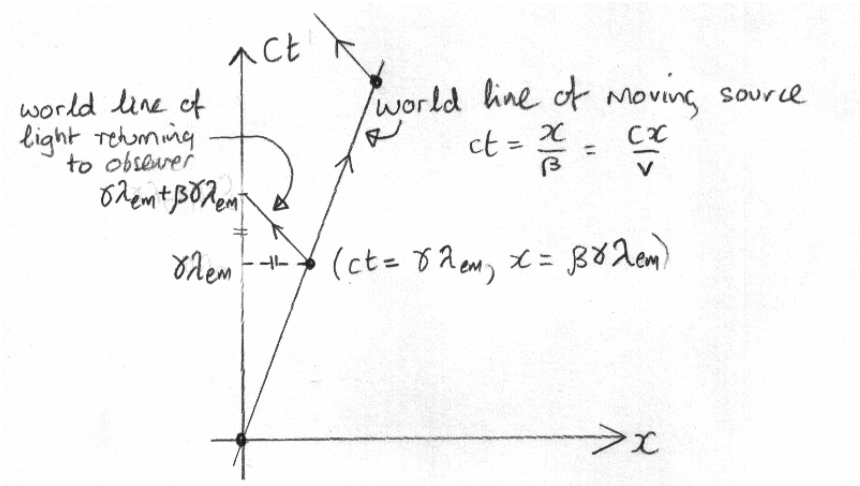

Figure 3: The world lines of the moving source and the returning light from the second of two successive wavefronts in the rest frame of the detector with respect to which the source is receding.

server with respect to whom the source is receding. Let us now draw a spacetime diagram in the reference frame of the light detector.

This diagram is exactly the same as Figure 1, except that the spacetime coordinates of all the events are scaled by a factor of γ. The reason for this scaling is that the moving source is effectively a clock and it is running slow due to the effect of time dilation. To the detector, however, the first wavefront arrives at $ct = 0$, and the second wavefront arrives at $ct = \lambda_{\mathrm{ob}}$, therefore we deduce that

$$\begin{aligned} \lambda_{\mathrm{ob}} &= \gamma\lambda_{\mathrm{em}} + \beta\gamma\lambda_{\mathrm{em}} \\ &= \lambda_{\mathrm{em}}\frac{1+\beta}{\sqrt{1-\beta^2}} \\ &= \lambda_{\mathrm{em}}\sqrt{\frac{(1+\beta)^2}{(1+\beta)(1-\beta)}} \\ &= \lambda_{\mathrm{em}}\sqrt{\frac{1+\beta}{1-\beta}}. \end{aligned} \tag{8}$$

Therefore the special relativistic redshift is given by

$$1 + z = \frac{\lambda_{\mathrm{ob}}}{\lambda_{\mathrm{em}}} = \sqrt{\frac{1+\beta}{1-\beta}}. \tag{9}$$

When β approaches 1, z tends to infinity.

4 The non-relativistic limit of the longitudinal relativistic doppler effect

Let us check that for small β, Equation 9 reduces to Equation 4. Where $\beta \ll 1$, we may write

$$\begin{aligned} 1 + z = \sqrt{\tfrac{1+\beta}{1-\beta}} &= (1+\beta)^{\frac{1}{2}}(1-\beta)^{-\frac{1}{2}} \\ &\simeq (1+\tfrac{\beta}{2})(1+\tfrac{\beta}{2}) \\ &\simeq 1+\beta, \end{aligned} \tag{10}$$

so that $z \simeq \beta$, the non-relativistic result.

5 The transverse relativistic doppler effect

Now let us consider the case where the source is moving at right angles to the line of sight to the observer. For example, the source might be positioned some way off on the x–axis, but have its motion parallel to the y–axis. Non-relativistically, we expect no doppler shift in the wavelength of the emitted light. However, since γ depends only on v^2, and is therefore independent of the direction of the motion of the clock, there is a transverse doppler shift even in this case, caused by time dilation of the source, If the period of the emitted wave is τ_{em} in its own rest frame, then in the rest frame of the detector, the period will be $\gamma\tau_{\mathrm{em}}$. Since the wavelength is c times the period, the observed wavelength is related to the emission wavelength by

$$\lambda_{\mathrm{ob}} = \gamma\lambda_{\mathrm{em}}. \tag{11}$$

Note that when $v \ll c$, $\gamma \simeq 1$, and there is zero transverse doppler shift.

6 The expanding Universe

In this section we attempt to head off a misconception that is particularly common amongst astronomy students that Equation 9 can be used to deduce the velocity of recession of faraway galaxies and clusters from the redshift of their spectral lines. Hubble and others studied the redshift of spectral lines from distant galaxies and in 1929 formulated Hubble's law, which states that the redshift of spectral lines from a source expanding with the universe is proportional to the distance to that source. However, the mechanism for the redshift is not the same as the mechanism which we have discussed above, though for objects at smaller distances the two descriptions do become equivalent. The cosmological redshifts of very faraway objects are best understood in terms of the expansion of the whole Universe, and the physics of this expansion is not as simple as doppler shift at the point of emission due to recession of the source.

To make this clear, let us discuss an example. Say at the moment of emission of the photons, their wavelength is 500 nm. Now suppose that in the intervening period between the photons being emitted from their source and arriving at Earth, the universe expanded by a factor of two (so that the light travel time between any given pair of galaxies doubled). Then the redshift of the photons would be exactly one, and their wavelength at arrival would be 1000 nm. However, it does not matter what the history of the scale factor of the Universe is between emission and reception. The scale factor could be increasing linearly, in which case the velocity of recession of the galaxies from each other would be a constant, and there is a nice analogy between Hubble expansion and simple recession. However, the scale factor could, for example, instead remain constant for the whole period, except for one discontinuous jump. The effect on the wavelength of the photons would be exactly the same, even though at the time of emission and absorption of the photons, the relative velocity of the source and emitter is zero. In this case, there is no simple way to understand the cosmological redshift in terms of the velocity of recession of the source.

If you are interested in cosmological redshifts, they are discussed both in PHY314 (relativity and cosmology) and PHY306 (introduction to cosmology), which you can take next year. The subject is also discussed very eloquently in 'Introduction to Cosmology' by Barbara Ryden, a very readable and well-written textbook which can be found in our library.

7 An example

GPS satellites orbit at a speed of 3.9 kilometers per second. How long does it take their onboard clocks to lose one second compared to a clock fixed on the Earth? Assume the effects of acceleration can be neglected, as can the velocity of the Earth's surface.

ANSWER: Assume the satellites motion is transverse to the Earth's surface. At a velocity of $3.9\,\mathrm{km/s}$, $\beta = 1.3 \times 10^{-5}$, so that $\gamma - 1 \simeq \beta^2/2$. For this discrepancy to cause a second of mismatch in N seconds, $N\beta^2/2 = 1$, so that $N = 2/\beta^2$, or $N = 1.2 \times 10^{10}\,\mathrm{s}$, or 375 years. In fact there are several other effects that cause differences between timekeeping of GPS clocks in orbiting satellites and terrestrial clocks, notably a general relativistic correction which is studied in PHY314.