Solutions to PHY206 relativity homework

1 (a) $E = 106\,\text{MeV}$

(b) $p = 106\,\text{MeV}/c$

(c) $\gamma = \dfrac{E}{mc^2} = \dfrac{106}{0.51} = 207.8$

(d) in lab $\begin{pmatrix} E/c \\ p \end{pmatrix} = \begin{pmatrix} 106\,\text{MeV}/c \\ -106\,\text{MeV}/c \end{pmatrix} = \begin{pmatrix} E/c \\ -E/c \end{pmatrix}$

in rest frame of electron

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ -E/c \end{pmatrix}$$

$$E'/c = \gamma E/c + \beta\gamma E/c$$

$$= \gamma(1+\beta) E/c$$

$$\approx 2\gamma E/c$$

$$E' = 2 \times 207.8 \times 106\,\text{MeV}$$

$$= 44\,\text{GeV}\,!!$$

So if the initial electron had been at rest in the lab you would need a 44 GeV positron beam to produce muons! It's far more efficient to run a colliding beam machine

2(a) $E_{\bar{p}} = 1.6\,\text{GeV}$ $\quad E_p = M_p c^2 = 0.94\,\text{GeV}$

$\bar{p} \longrightarrow p$

$E_{\gamma 2} \leftsquigarrow \quad \rightsquigarrow E_{\gamma 1}$

momentum $-\dfrac{E_{\gamma 2}}{c}$ $\quad$ momentum $+\dfrac{E_{\gamma 1}}{c}$

conserve energy $\quad E_{\bar{p}} + E_p = E_{\gamma 1} + E_{\gamma 2}$

conserve momentum $\quad \sqrt{E_{\bar{p}}^2 - m_p^2 c^4} = E_{\gamma 1} - E_{\gamma 2}$

$$2E_{\gamma 1} = E_{\bar{p}} + E_p + \sqrt{E_{\bar{p}}^2 - m_p^2 c^4}$$

$$E_{\gamma 1} = \frac{1}{2}\left(E_{\bar{p}} + m_p^2 c^4 + \sqrt{E_{\bar{p}}^2 - m_p^2 c^4}\right)$$

$$= \frac{1}{2}\left(1.6\,\text{GeV} + 0.94\,\text{GeV} + \sqrt{1.6^2 - 0.94^2}\right)$$

$$E_{\gamma 1} = 1.92\,\text{GeV}$$

$$E_{\gamma 2} = E_p + E_{\bar{p}} - E_{\gamma 1} = 0.62\,\text{GeV}$$

2b) There are two ways to do this problem

METHOD 1.) ~~Realise that in the rest frame of the~~ antiproton, the proton is approaching with a total energy of 1.6 GeV, so that the outgoing energies of photons 1, & 2, are interchanged so

$$E_{\gamma_1}' = E_{\gamma_2} = 0.62\,\text{GeV}$$

$$E_{\gamma_2}' = E_{\gamma_1} = 1.92\,\text{GeV}$$

METHOD 2.) Lorentz transformation

Figure out β and γ for the incoming antiproton in the lab

$$\gamma = \frac{E_{\bar{p}}}{M_{\bar{p}}c^2} = \frac{1.6}{0.94} = 1.7$$

$$\gamma^2 = \frac{1}{1-\beta^2} \quad \sqrt{1-\frac{1}{\gamma^2}} = \beta = 0.81$$

$$\left.\begin{aligned} E_{\gamma_1} &= 1.92\,\text{GeV} \\ p_{\gamma_1}c &= 1.92\,\text{GeV} \end{aligned}\right\}$$ Because for a <u>photon</u> $E = cp$

$$E_{\gamma_1}' = \gamma(E_{\gamma_1} - \beta p_{\gamma_1}c)$$

$$= 1.7 \times 1.92\,\text{GeV} \times (1 - 0.81)$$

$$= 0.62\,\text{GeV}$$

$$E_{\gamma_2} = 0.62\,\text{GeV}$$

$$p_{\gamma_2}c = -0.62\,\text{GeV}$$ '−' sign because its moving to the left

$$E_{\gamma_2}' = \gamma(E_{\gamma_2} - \beta p_{\gamma_2}c)$$

$$= 1.7 \times 0.62\,\text{GeV} \times (1 + 0.81)$$

$$= 1.92\,\text{GeV}$$

Example exam

2. π^0–mesons decay to two gamma rays via $\pi^0 \rightarrow 2\gamma$. The rest mass of a π^0 meson is $(135.1\,\text{MeV})/c^2$. In a particular lab experiment, the π^0 mesons have energies in the range 6 GeV to 18 GeV. Take the mean life τ_0 of a π^0–meson to be 2.9×10^{-16} s. For this question, use $c = 2.9974 \times 10^8\,\text{m}\,\text{s}^{-1}$.

   (a) What are the lowest and highest possible velocities in the π^0–meson beam? [2]

   (b) What are the lowest and highest possible π^0–meson mean lifetimes as measured in the lab, in seconds? [2]

   (c) For π^0–mesons at the lower and upper edges of the energy range, what distances between production and decay do the mean lives correspond to, in micrometres? Is it possible for a π^0 to survive have a longer range between production and decay than the greater of these two numbers? [2]

   (d) In the lab, what is the maximum possible energy achievable by a photon from a π^0–meson decay in this particular beam? [4]

a)

$$\gamma_H = \frac{18\,[\text{GeV}]}{135\,[\text{MeV}]} = 133$$

$$\gamma_L = \frac{6\,[\text{GeV}]}{135\,[\text{MeV}]} = 44.4$$

Both >> 1 so velocity very similar to c

max velocity $c - \varepsilon_H$ where $\gamma_H \simeq \sqrt{\frac{c}{2\varepsilon_H}}$

$$\frac{c}{2\varepsilon_H} = 133^2 \qquad \varepsilon_H = \frac{3\times10^8\,[\text{ms}^{-1}]}{(2\times133^2)} = 8.5\,\text{kms}^{-1}$$

$$V_{max} = c - 8.5\,\text{km s}^{-1}$$

min velocity $c - \varepsilon_L$ where $\gamma_L = \sqrt{\frac{c}{2\varepsilon_L}}$

$$\frac{c}{2\varepsilon_L} = 44.4^2 \qquad \varepsilon_L = \frac{3\times10^8\,[\text{ms}^{-1}]}{2\times44.4^2} = 76\,\text{km s}^{-1}$$

$$V_{min} = c - 76\,\text{km s}^{-1}$$

b) $\tau_{max} = \gamma_H \tau_0$

$= 133 \times 2.9 \times 10^{-16}\,s$

$= 3.9 \times 10^{-14}\,s$

$\tau_{min} = \gamma_L \tau_0$

$= 44.4\, \tau_0$

$= 1.28 \times 10^{-14}\,s$

c) Distances are

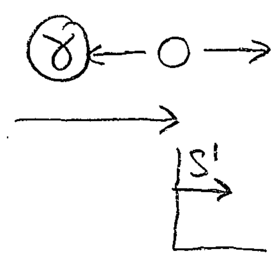
$c\tau_{max} = 3\times10^{8}_{ms^{-1}} \times 3.9\times10^{-14}\,s$

$= 11.7\,\mu m$

$c\tau_{min} = 3\times10^{8}\,[ms^{-1}] \times 1.28\times10^{-14}\,s$

$= 3.8\,\mu m$

d) Max possible energy when γ ~~~~ emitted forward.



$E_\gamma^{CM} = \frac{135.1}{2}$ MeV $\qquad \gamma = 133$

$\beta = 1$

$cp_\gamma^{CM} = -\frac{135.1}{2}$ MeV

$E_\gamma' = 133 \times \left(\frac{135.1}{2} + \frac{135.1}{2}\right)$

$= 18.0$ GeV

A man carrying a pole horizontally runs at a constant speed into a barn whose long axis is parallel to the pole. The barn is 20 metres in length. The rest length of the pole is 25 metres. Once the rear end of the pole enters the barn, the rear door is shut behind the pole. Although the pole at rest is longer than the barn, and therefore when at rest it won't fit, the runner gets around this difficulty by running so fast that the pole's length is Lorentz contracted, and the pole does fit into the barn. For this problem, consider the runner to be travelling in the +x–direction.

1. At what fraction of c must the runner sprint so that the pole is Lorentz contracted to a length of 15 metres to an observer in the reference frame of the barn? What are the values of the special relativistic β and γ corresponding to this speed?

2. In the (unprimed) reference frame of the barn, considering the event of the rear of the pole entering the barn to have coordinates $(ct = 0, x = 0)$, what are the coordinates of the event of the front of the pole touching the far wall of the barn?

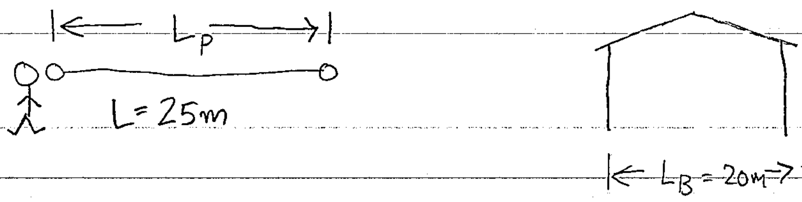
3. Consider a second (primed) observer in the reference frame of the runner. In this frame, calculate the coordinates (ct', x') of:

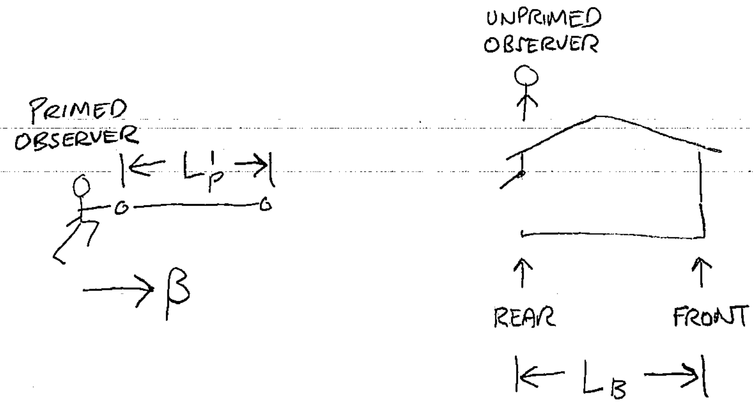
   (a) The rear of the pole entering the rear of the barn;

   (b) The front of the pole touching the front wall of the barn.

4. In the reference frame of the runner, once the front of the pole hits the front of the barn, how much longer does it take until the rear of the pole enters the barn door? Show that the rear end of the pole doesn't receive information that the front end has hit the far wall until after it has entered the barn. Can the pole be perfectly rigid? Explain your answer.

BARN DOOR PARADOX QUESTION



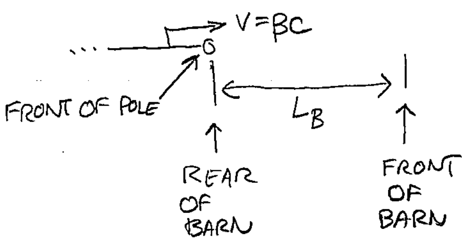


$\boxed{\text{Lorentz contraction of the pole}}$

a) $L'_P = \dfrac{L_P}{\gamma} \Rightarrow \gamma = \dfrac{L_P}{L'_P} = \dfrac{25\,[m]}{15\,[m]} = \dfrac{5}{3}$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \frac{4}{5}.$$

b) $\boxed{\text{In the unprimed frame, the rest frame of the barn}}$



Spacetime coordinates for front of pole entering rear of barn

$$(ct, x) = (0, 0)\,[m]$$

Time front of pole takes to traverse barn and reach front of barn

$$t_B = \frac{L_B}{V} = \frac{L_B}{\beta c} \Rightarrow ct_B = \frac{L_B}{\beta}$$

$$x_B = L_B \qquad \text{so} \quad (ct_B, x_B) = \left(\frac{L_B}{\beta}, L_B\right)$$

$$= (25\,[m],\ 20\,[m])$$

c) $\boxed{\text{In the primed frame, the rest frame of the runner}}$

Spacetime coordinates for the front of the pole entering the rear of the barn

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Spacetime coordinates for front of pole hitting front of barn.

$$\begin{pmatrix} ct_B' \\ x_B' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} ct_B \\ x_B \end{pmatrix} = \begin{pmatrix} 5/3 & -4/3 \\ -4/3 & 5/3 \end{pmatrix}\begin{pmatrix} 25 \\ 20 \end{pmatrix}[m] = \begin{pmatrix} 15 \\ 0 \end{pmatrix}[m]$$

When does the runner enter the barn

in rest frame of runner

in Barn frame

$$ct_R = \frac{L_P'}{\beta} = \frac{15\,m}{\frac{4}{5}} = 18\cdot75$$

$$x_R = 0$$

$$\begin{pmatrix} ct_R' \\ x_R' \end{pmatrix} = \begin{pmatrix} 5/3 & -4/3 \\ -4/3 & 5/3 \end{pmatrix} \begin{pmatrix} 18\cdot75 \\ 0 \end{pmatrix} = \begin{pmatrix} 31\cdot25 \\ -25 \end{pmatrix} [m]$$

in rest frame of runner, time between front of pole hitting front of barn and rear of pole (and runner) entering rear of barn is

$$\frac{31\cdot25\ m - 15\ m}{c} = 54\,ns$$

in rest frame of runner, pole is 25 m long

time for a light signal to traverse the pole is

$$\Delta t_{LS} = \frac{25\ m}{3\times10^8\,ms^{-1}} = 83\,ns$$

so a light signal cannot propagate from one end of the pole to the other in the time between the front of the pole striking the front of the barn and the runner and back of the pole entering the barn door

In rest frame of runner