By considering the path of a photon, show that the *comoving proper distance* between an object and the origin is given by

$$r = c\int_{t_e}^{t_o} \frac{\mathrm{d}t}{a(t)},$$

where t_e is the time of emission and t_o is the time of observation. [2]

The RW metric is $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + x_k^2\mathrm{d}\Omega^2\right)$. A photon has $\mathrm{d}s^2 = 0$, because it travels at the speed of light (and therefore in a static universe we must have $\mathrm{d}r/\mathrm{d}t = c$) [1]
so for a radial trajectory the RW metric gives $c^2\,\mathrm{d}t^2 = a^2(t)\,\mathrm{d}r^2$, from which result follows immediately. [1]

Hence show that, in an expanding universe, the observed light will be redshifted such that

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_o}{a(t_o)},$$

where λ_e is the emitted wavelength and λ_o is the observed wavelength. [4]

If a wave crest is emitted at time t_e and observed at time t_o, then the comoving proper distance r is given by $r = c\int_{t_e}^{t_o}\mathrm{d}t/a(t)$ as in the question. Considering the emission and reception of the *next* wave crest, which is emitted at $t_e + \lambda_e/c$ and observed at $t_o + \lambda_o/c$, gives $r = c\int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c}\mathrm{d}t/a(t)$. [1]

Now subtract from both integrals the common interval $c\int_{t_e+\lambda_e/c}^{t_o}\mathrm{d}t/a(t)$, giving

$$c\int_{t_e}^{t_e+\lambda_e/c} \frac{\mathrm{d}t}{a(t)} = c\int_{t_o}^{t_o+\lambda_o/c} \frac{\mathrm{d}t}{a(t)}.$$ [1]

But in this small time interval (assuming, reasonably, that $\lambda << c/H_0$) we can neglect the change in a, so this equation becomes

$$\frac{c}{a(t_e)}\int_{t_e}^{t_e+\lambda_e/c} \mathrm{d}t = \frac{c}{a(t_o)}\int_{t_o}^{t_o+\lambda_o/c} \mathrm{d}t$$

which integrates trivially to give the required answer. [2]