The University Of Sheffield.

Department
Of
Physics &
Astronomy

PHY226
Mathematical Methods
for Physics and Astronomy

Topics 1-4

Autumn Semester 2010

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PHY226 - Mathematical Methods for Physics and Astronomy

Taught in three serial blocks by different lecturers

Topics 1-4
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1. Revision of algebraic methods
2. Complex numbers and functions
3. Ordinary differential equations
4. Fourier series

Topics 5-6
Prof. David Mowbray, E14, d.mowbray@sheffield.ac.uk

5. Fourier integrals
6. Convolution theorem

Topics 6-9
Dr Vitaly Kudryavtsev, F9b, v.kudryavtsev@sheffield.ac.uk

6. Partial differential equations
7. Solution of the diffusion equation
8. Solution of partial differential equations in three dimensions
9. Spherical co-ordinates and brief introduction to spherical harmonics

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This course is intended to equip you with the mathematical understanding and skills that you need for your future studies of physics and astronomy. The selected topics are the really useful ones – mathematics which crops up time and time again.

Reference material

Lectures will impart some *knowledge* of mathematics but *skill* can only be obtained by practising! Examples will be given in each lecture but a massive collection of further worked **online problems** can be found at **http://www.hep.shef.ac.uk/phy226.htm**

The **notes** are pretty complete and can also be found online so there are no compulsory textbooks. *Jordan & Smith* covers approximately the first half of the course and is recommended. Two other excellent textbooks both of which cover most of the course are:

Mary L. Boas – Mathematical Methods in the Physical Sciences (Wiley)
Erwin Kreyszig – Advanced Engineering Mathematics (Academic Press)

The **course pack** is a set of photocopied pages from various textbooks and is provided as reference material for those parts of the course *not* covered by Jordan & Smith (mainly partial differential equations). It contains theory and worked examples, and is intended to reinforce the lectures. The packs can be bought for £7 from the departmental office, although we will always have 3 copies that can be loaned out on a short-term basis.

The Physics and Astronomy **Formula and Data sheet** is also a really useful point of reference. You should always have a copy when doing problems or reading the notes.

Assessment

Two homeworks	20% (10% each)
Attendance and performance at two problem classes	10% total
Final examination	70% total

Homework must be handed to the E floor Departmental Office no later than the deadline dates.
Problem class questions should be handed to problem class leader.

Content of Lectures 1-8

Topic 1: Revision of Algebra

1.1 Multiplying Brackets

All terms are included, e.g.

$(x+y)^3 = (x+y)(x^2+2xy+y^2) = x(x^2+2xy+y^2) + y(x^2+2xy+y^2)$

Also $(x+a)(x-a) = x^2 - a^2$

You can check these by choosing a simple value for *x and y* in the above expression.

1.2 Binomial Series (See exam data sheet)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \binom{n}{k}x^n + \ldots, \text{ where } \binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

- When *n* is a positive integer we have a finite series: i.e. the series terminates.
- When *n* is negative or non-integer, the series does *not* terminate.
- The series converges for all $|x| < 1$ since *x* to any power will be smaller than *x*.

The most useful job of both the binomial and Taylor series is to intelligently approximate an impossibly complicated expression to a few simple terms that pretty well equal the full solution. We do this when we say 'taking only the first two or three terms'. However we need to apply it correctly.

We can only approximate $(1+x)^n \approx 1 + nx$ or $(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2$ **when $x \ll 1$.**

Consider $(a+b)^n$. This can be rewritten $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n(1+x)^n$ where $x = b/a$

The intelligent part is to choose *a, b* such that $|a| > |b|$ so $|x| \ll 1$.

How many terms you need to use depends on how small *x* is and on how accurately you need the answer. As a rule of thumb, in most physical applications it is fine to use these approximations for $x < 0.1$. Obviously if *n* is very large you need a correspondingly smaller value of *x* for rapid convergence.

Example 1.1

(a) Expand $\frac{1}{1+x}$ *in powers of x.*

$$\frac{1}{1+x} = (1+x)^{-1} \qquad n=-1$$

$$\frac{1}{1+x} = 1 + (-1)x + \frac{(-1)(-2)x^2}{2} + \frac{(-1)(-2)(-3)x^3}{3.2} = 1 - x + x^2 - x^3 + \ldots$$

(b) If $x = 0.1$ *and we need accuracy to about* 1%*, how many terms do we need?*
Since $x^3 \sim 10^{-3}$ we would guess we could stop at x^2. Let's check this:
The exact answer is $(1+0.1)^{-1}$ = 0.9090909… . To 1% accuracy this is 0.91.
The expansion above gave us:- first term = 1, second term = $-x$, third term = x^2, fourth term = $-x^3$ So $(1+0.1)^{-1}$ = 1 – 0.1 + (0.1^2) – (0.1^3). Yes we can stop after two terms.

Example 1.2 Rewrite $(\sin\theta + \cos\theta)^{15}$ for small θ using the binomial expansion for the first three terms.

$$(\sin\theta + \cos\theta)^{15} = (\cos\theta(\tan\theta + 1))^{15} = \cos^{15}\theta(1+\tan\theta)^{15}$$

$$(1+\tan\theta)^{15} = 1 + 15\tan\theta + \frac{(15)(14)}{2}\tan^2\theta + \ldots$$

$$(1+\tan\theta)^{15} = 1 + 15\tan\theta + 105\tan^2\theta + \ldots$$

$$\cos^{15}(1+\tan\theta)^{15} = \cos^{15}\theta + 15\cos^{15}\theta\tan\theta + 105\cos^{15}\theta\tan^2\theta + \ldots$$

NB. The binomial expansion works for any value of x and n.

It's just more useful if x <<1

1.3 **Taylor Series** (also on the exam data sheet)

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \ldots + \frac{f^n(a)}{n!}x^n + \ldots \qquad \text{where} \qquad f^n(a) = \left.\frac{d^n f}{dx^n}\right|_{x=a}$$

Unlike the binomial expansion, the Taylor expansion can also be used for *any* function that has a derivative. Note that if $a = 0$ then the Taylor expansion is known as a Maclaurin expansion.

The range of x for which there is convergence depends on the function f. But in practice the series is only *useful* if the first couple of terms give an adequate approximation, which means we need $x<<1$.

Examples 1.3 and 1.4

1.3 Find the Maclaurin expansion of $e^{\alpha x}$?

$$f(x) = e^{\alpha x}$$
$$f'(x) = \alpha e^{\alpha x}$$
$$f''(x) = \alpha^2 e^{\alpha x}$$

$$f(x) = e^{\alpha x} = e^0 + \alpha e^0 x + \frac{\alpha^2 e^0 x^2}{2!} + \frac{\alpha^3 e^0 x^3}{3!} +$$

$$e^{\alpha x} = 1 + \alpha x + \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{6} + \ldots$$

1.4 Find the first two terms of the expansion of $\tan(\frac{\pi}{4} + x)$.

$$\tan\left(\frac{\pi}{4} + x\right) = \tan\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right)x = 1 + 2x$$

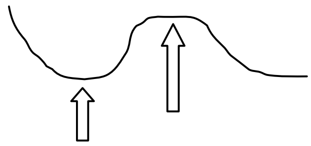
$$f\left(\frac{\pi}{4}\right) = 1$$

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$$

1.4 Why do we use these expansions so often in physics?

Because we like to solve *easy* problems.
E.g. Stability

Suppose a particle of mass *m* lies on a potential surface $V(x)$ at $x = x_0$. There is no resultant force on it at this point – i.e. the particle is in equilibrium. This would be true if x_0 were at either of the positions marked with arrows in the figure.

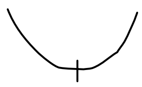


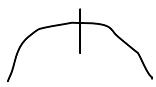
The condition that there is no force at x_0 is that $\left.\frac{dV(x)}{dx}\right|_{x=x_0} = 0$.

But we want to know whether the equilibrium is stable!

We find this by asking if the potential increases or decreases as we move away from x_0.

This is equivalent to determining whether $\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0}$ is positive or negative.

 $\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0} > 0$ Stable

 $\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0} < 0$ Unstable

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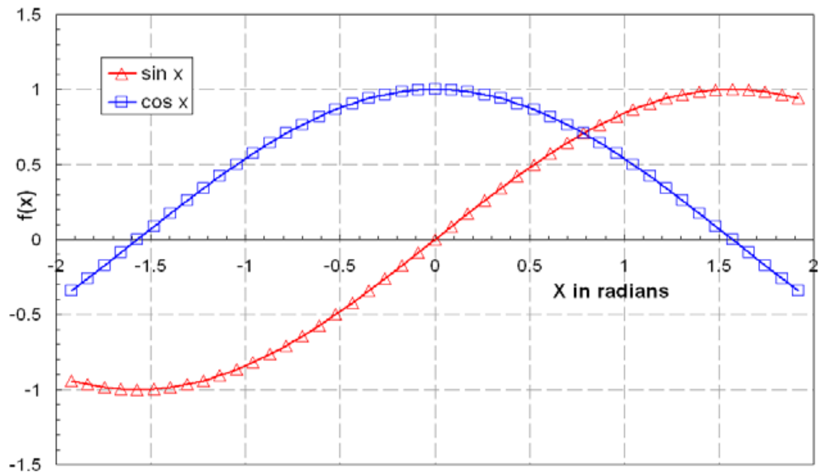
Finding and differentiating a complete expression for $V(x)$ might be very hard, but an approximate expression valid near x_0 is all we need to answer our question.

1.5 Trigonometric and Hyperbolic functions

A key observation looking at sin(x) and cos(x) below is that cos(x) is symmetrical in the y-axis whereas sin(x) is not. This can be written mathematically as:

cos(x) = cos(-x) and sin(x) = - sin(-x)

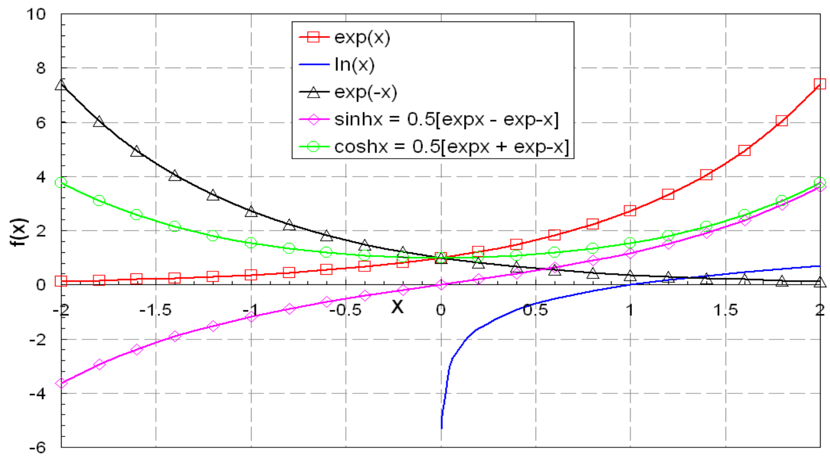
These relationships are crucial in this course



We can combine exponentials into the *hyberbolic* functions:

$$\cosh \alpha x = \frac{1}{2}\left[e^{\alpha x} + e^{-\alpha x}\right]$$ is an *even function*: $\cosh \alpha x = \cosh(-\alpha x)$

and $$\sinh \alpha x = \frac{1}{2}\left[e^{\alpha x} - e^{-\alpha x}\right]$$ is an *odd function*: $\sinh \alpha x = -\sinh(-\alpha x)$.



Using the definition of the derivative it is easy to show that:

$$\frac{d\cosh \alpha x}{dx} = \frac{1}{2}\left[\frac{de^{\alpha x}}{dx} + \frac{de^{-\alpha x}}{dx}\right] = \frac{\alpha}{2}\left[e^{\alpha x} - e^{-\alpha x}\right] = \alpha \sinh \alpha x$$

Similarly $$\frac{d\sinh \alpha x}{dx} = \alpha \cosh \alpha x .$$

1.6 **Exponentials** are powers and so they satisfy:

$e^{a+b} = e^a e^b$ and $e^{-a} = 1/e^b$

1.7 Natural logarithms are defined by

$$y = e^x \qquad x = \ln y \, .$$

We also have that

$$\ln y_1 - \ln y_2 = \ln(y_1 / y_2) \quad \text{and} \quad \ln y_1 + \ln y_2 = \ln(y_1 y_2)$$

We can relate natural logs to those to base 10:

Define $w = \log_{10} y$. This expression *means* that $y = 10^w$.

Take natural logs of both sides:

$$\ln y = \ln\left(10^w\right) = w \ln\left(10\right) \qquad w = \frac{\ln y}{\ln(10)} \quad \text{or} \quad y = e^{w \ln(10)} \, .$$

Online Problems (Topic 1, questions 1-5)

1. Consider $(52+3)^{1.5}$. Your calculator will give you an answer of 407.8909 to 4dp. How many terms of a binomial expansion do you need to get within 1% of this?

2. In question 1 what would happen if, by factorising differently, you had chosen to expand $\left(1 + \tfrac{52}{3}\right)^{1.5}$ instead of $\left(1 + \tfrac{3}{52}\right)^{1.5}$?

3. Expand $(a+b)^5$, i) where a and b are 2 and 10 respectively and ii) where a and b are 10 and 2 respectively

4. Expand $\left(a^2 + b^2\right)^{-1/4}$ in powers of b/a up to terms of order b^4

5. Expand $\ln(1+x)$ via the Taylor expansion up to the 4th term

6. Topic 2. Complex Numbers

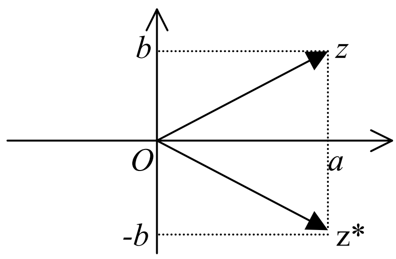
2.1 Argand diagram

Let $z = a + ib$ where $i^2 = -1$

To represent this number on an **Argand diagram**, plot the point with Cartesian coordinates (*a*, *b*). i.e. real numbers run along the *x* axis and imaginary numbers along the *y* axis.

The **complex conjugate** is $z^* = a - ib$

(Note: Physicists usually use i, engineers often use j and physicists *always* denote complex conjugates by z* *not* $\bar{z}$)



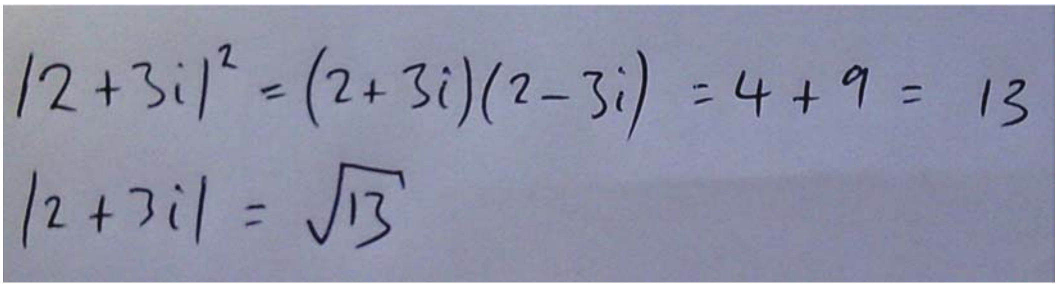
By Pythagoras, the length OZ is $r = \sqrt{a^2 + b^2} = |z|$.
Note that this length is also equal to $\sqrt{zz^*}$

$$zz^* = (a + ib)(a - ib) = a^2 + (ib)(-ib) = a^2 + b^2 .$$

We also write $a^2 + b^2 = zz^* = |z|^2$

where $|z| = \sqrt{zz^*}$ and is called the modulus of *z*.

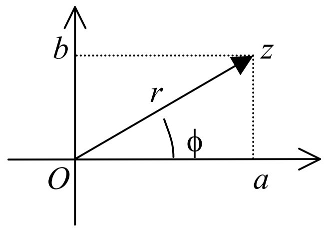
Example 2.1 Find the modulus of $|(2 + 3i)|$?



2.2 Polar form

We can also write

$z = re^{i\phi} = r(\cos\phi + i\sin\phi)$ where $0 < \phi < 2\pi$



r = is again called the modulus,
ϕ is called the *argument* or *phase*.

For a proof of this relationship see Lecture 2, problem 2 in the online Problems.

Then $z^* = re^{-i\phi} = r(\cos\phi - i\sin\phi)$
So $zz^* = r^2 e^{i\phi}e^{-i\phi} = r^2$ since $e^{i\phi}e^{-i\phi} = e^{i(\phi-\phi)} = 1$
Clearly $a = r\cos\phi$, $b = r\sin\phi$ and $r = \sqrt{a^2 + b^2} = |z|$

Note that $|z|$ and $zz^* = |z|^2$, are always *real*, whereas $z^2 = a^2 + 2iab - b^2 = r^2 e^{2i\phi} \neq |z|^2$ is usually *complex*. In physics we *always* need to get real answers, hence in quantum mechanics etc. one takes $|\psi|^2$ not ψ^2. (In optics and E&M you may sometimes take the real part to get your answer.)

Changing between the forms $z = a + ib$ and $z = re^{i\phi}$

You are strongly advised to first *plot the number on an Argand diagram*. Without this it is easy to make mistakes about minus signs and angles, etc.!

Given $z = a + ib$, to find the form $z = re^{i\phi}$
Find r using $r = \sqrt{a^2 + b^2}$.
Find ϕ using $\tan\phi = \dfrac{b}{a}$ (specifying its sign from the quadrant of the Argand diagram.)
Given $z = re^{i\phi}$, to find the form $z = a + ib$ is easier: $a = r\cos\phi$ and $b = r\sin\phi$.

Why do we need both forms?
It is easier to *add* and *subtract* complex numbers in the form $z = a + ib$ but easier to *multiply, divide,* take *powers* and *roots* when they are in the form $z=re^{i\phi}$.
In physics we almost always use the form $z = re^{i\phi}$.

Addition and Subtraction

If $z = a + ib$ and $w = c + id$

then $z + w = (a + c) + i(b + d)$ and $z - w = (a - c) + i(b - d)$.

Multiplication and Division

For this we *always* use the form $z = r\,\mathrm{e}^{\,\mathrm{i}\phi}$

Let $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$
then $z_1 z_2 = r_1 e^{i\phi_1} r_2 e^{i\phi_2} = r_1 r_2 e^{i(\phi_1 + \phi_2)}$.

i.e. *multiply* the moduli and *add* the arguments (phases).

Similarly for division:

$(z_1 / z_2) = (r_1 / r_2)\, e^{i(\phi_1 - \phi_2)}$..
i.e. we *divide* the moduli and *subtract* the arguments.

Example 2.2 Express (1 + i) ÷ (1 + 1.73i) in polar coordinates?

$z_1 = 1 + i = \sqrt{2}\, e^{i\pi/4}$

$\tan\theta = \frac{1}{1}$ $r = \sqrt{2}$

$\theta = \pi/4$

$z_2 = 1 + 1.73i$

$\tan\theta = 1.73$ $r = \sqrt{1+3} = 2$

$\theta = \pi/3$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\pi/4}}{2\, e^{i\pi/3}} = \frac{\sqrt{2}}{2} e^{i(\pi/4 - \pi/3)}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2} e^{-i\pi/12} = 0.707 e^{-i\pi/12}$$

2.3 Powers and Roots

Again we *always* use the polar form. For a real number power it is straightforward:

$z^n = r^n e^{in\phi}$

i.e. we take the modulus to the n^{th} power and multiply the argument (or phase) by *n*.

Roots are trickier. We defined ϕ to lie in the region $0 < \phi < 2\pi$. But this will need to be extended if we want to get *all* the roots of a complex number.

We define $z = re^{i(\phi + 2p\pi)}$ where *p* is an integer.

To find an n^{th} root, we need to take *n* distinct values of *p*: $p = 0,\ p = 1, p = 2,\ \ldots,\ p = n - 1$.

Then there are *n* distinct roots to $z^{1/n} = r^{1/n} e^{i(\phi + 2p\pi)/n}$.

<u>Example 2.3</u> : If $z = 9\, e^{i\pi/3}$ what is $z^{½}$?

Step 1:

write down z in polars with the $2\pi p$ bit added on to the argument. $z = 9e^{i\,(\pi/3 + 2\pi p)}$

Step 2:

say how many values of p you'll need and write out the rooted expression here $n = 2$ so I'll need 2 values of p; $p = 0$ and $p = 1$ $z^{½} = \sqrt{9}\, e^{i\,(\pi/3 + 2\pi p)/2}$

Step 3:

Work it out for each value of p: $z^{½} = 3e^{i\,(\pi/3)/2} = 3e^{i\,(\pi/6)}$ for $p = 0$

$z^{½} = 3e^{i\,(\pi/3 + 2\pi)/2} = 3e^{i\,(\pi/6 + \pi)}$ for $p = 1$

There are your answers but remember that $e^{i\phi} = (cos\phi + isin\phi)$ so $e^{i\pi} = -1$

It's therefore better to write $z^{½} = 3e^{i\,(\pi/6 + \pi)} = 3e^{i\,\pi/6}(e^{i\pi}) = -3e^{i\,\pi/6}$ for $p = 1$, and $3e^{i\,(\pi/6)}$ for $p = 0$

Example 2.4: If $z = 27\,e^{i\pi/2}$ what is $z^{1/3}$?

***Step 1*:** write down z in polars with the $2\pi p$ bit added on to the argument.

***Step 2*:** say how many values of p you'll need and write out the rooted expression

***Step 3*:** Work it out for each value of p

Step 1 : $z = 27e^{i(\pi/2 + 2\pi p)}$

Step 2 : for $n=3$ $p = 0, 1, 2$

Step 3 : for $p=0$ $z^{1/3} = 3e^{i(\pi/2)/3} = 3e^{i\pi/6}$

for $p=1$ $z^{1/3} = 3e^{i(\pi/2 + 2\pi)/3} = 3e^{i(5\pi/6)}$

for $p=2$ $z^{1/3} = 3e^{i(\pi/2 + 4\pi)/3} = 3e^{i(9\pi/6)}$

So the 3 roots of $z^{1/3}$ are : $3e^{i\pi/6}$, $3e^{i5\pi/6}$, $3e^{i9\pi/6}$

2.4 Exponentials and Trigonometric functions

$e^{ikx} = \cos kx + i\sin kx$; and $e^{-ikx} = \cos kx - i\sin kx$

Rearranging gives $\cos kx = \frac{1}{2}(e^{ikx} + e^{-ikx});$ $\sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx})$

<u>This is a key observation…remember this.</u>

2.5 Differentiation of a Complex Exponential

We know $\frac{d}{dx}e^{kx} = ke^{kx}$. Since i is just a constant, we similarly have $\frac{d}{dx}e^{ikx} = ike^{ikx}$

Note that is much nicer to differentiate exponentials than sines and cosines because we get exactly the same function as we started with, just multiplied by a constant.

Online problems for (Topic 2, questions 1-11)

1. Expand $\sinh x$ using the Taylor series up to the 5th power term in x

2. Prove the relationship $z = re^{i\phi} = r(\cos\phi + i\sin\phi)$ using the Taylor expansion

3. Find the modulus of i) $6+7i$ ii) $\frac{6+7i}{3+2i}$ and iii) plot $(6+7i)-(3+2i)$ on an Argand diagram and give the answer in polar co-ordinates. iv) Express $z = 5e^{i\pi/6}$ in Cartesian co-ordinates and v) Convert $3+4i$ and $5+12i$ into polar co-ordinates and multiply them.

4. In quantum mechanics we sometimes need to evaluate the modulus squared of the sum of two complex numbers. If $z_1 = Ae^{i\alpha}$ and $z_2 = Be^{-i\beta}$ find $|z_1 + z_2|^2$ Manipulate your answer into the form $A^2 + B^2 + AB\cos(\alpha - \beta)$

5. $z = 2+3i$ By changing into polar co-ordinates find z^{18}?

6. $z = 16e^{i\pi/4}$ Find $z^{1/4}$?

7. Show that $\left|\frac{a+ib}{a-ib}\right|^2 = 1$ for any real numbers a and b

8. Find $\sqrt{i}$

9. $z = 1+i$ Find $z^{1/2}$

10. Prove $x = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$ can be written as $x = e^{\alpha t}(C\sin\beta t + D\cos\beta t)$ and define C and D in terms of A and B

11. Prove that $x = e^{\alpha t}(C\sin\beta t + D\cos\beta t)$ can be written as $x = Ee^{\alpha t}(\cos(\beta t + \phi))$ and define ϕ and E in terms of C and D.

Topic 3. Ordinary Differential Equations (ODE's)

Most of physics involves the solution of differential equations! The solution of *ordinary* differential equations (ODEs) was covered in PHY112. 'Ordinary' means that all functions are of only one variable. We will revise the theory and explore some examples, especially harmonic oscillators. Later lectures will address the solution of *partial* differential equations featuring multiple variables.

3.1 First Order ODEs (i.e. 1 variable and no higher than $\frac{dx}{dt}$ terms)

Revision of Theory

You should be aware of two possible methods for solving 1st order ODEs. Which method you use depends on the equation you are trying to solve.

1. Some equations can be solved by the method of **separation of the variables**: rearrange the equation so that each side involves only one variable, then integrate both sides.
2. The method of **trial solution** may be used.

The **general solution** of a 1st order equation will contain <u>one *arbitrary constant*</u>; the value of the constant is determined by the *boundary conditions*, yielding a **particular solution**.

<u>Example</u>: Radioactive Decay

Consider a sample of radioactive material. Let N be the number of undecayed atoms at time *t*. At any time, the rate at which atoms decay is proportional to N.

I.e. $\frac{dN(t)}{dt} = -\lambda N(t)$ where λ is the decay constant. Given that $N = N_0$ at $t = 0$, find an expression for N at later times.

Method 1

a) $\frac{dN}{N} = -\lambda dt$ can be rearranged and both sides integrated: $\int \frac{dN}{N} = -\lambda \int dt$.

Performing these (indefinite) integrals we obtain $\ln N = -\lambda t + c$ (*remember c !)*

Hence $N = e^{-\lambda t + c} = e^{-\lambda t} e^{c} = Ae^{-\lambda t}$ where $A = e^{c}$.

Using the boundary condition that at $t = 0, N = N_0$, we find $A = N_0$. Hence $N(t) = N_0\, e^{-\lambda t}$.

b) Alternatively the boundary condition information can be entered as the limits of definite integrals:

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_0^t dt \quad \text{giving} \quad \ln N - \ln N_0 = \ln \frac{N}{N_0} = -\lambda t \quad \text{, hence } N(t) = N_0\, e^{-\lambda t}.$$

Method 2

We may guess that the equation has a solution of the form $N(t) = Ae^{mt}$.

Substituting this trial solution into the equation gives $\frac{dN(t)}{dt} = mN(t) = -\lambda N(t)$.

So it is a solution if $m = -\lambda$. i.e. the general solution is $N(t) = Ae^{-\lambda t}$.

Applying the boundary condition we find the solution as before.

Example 3.1 The growth of an ant colony is proportional to the number of ants. If at t = 0 days there are only 2 ants, but after 20 days there are 15 ants, what is the differential equation and what is its solution?

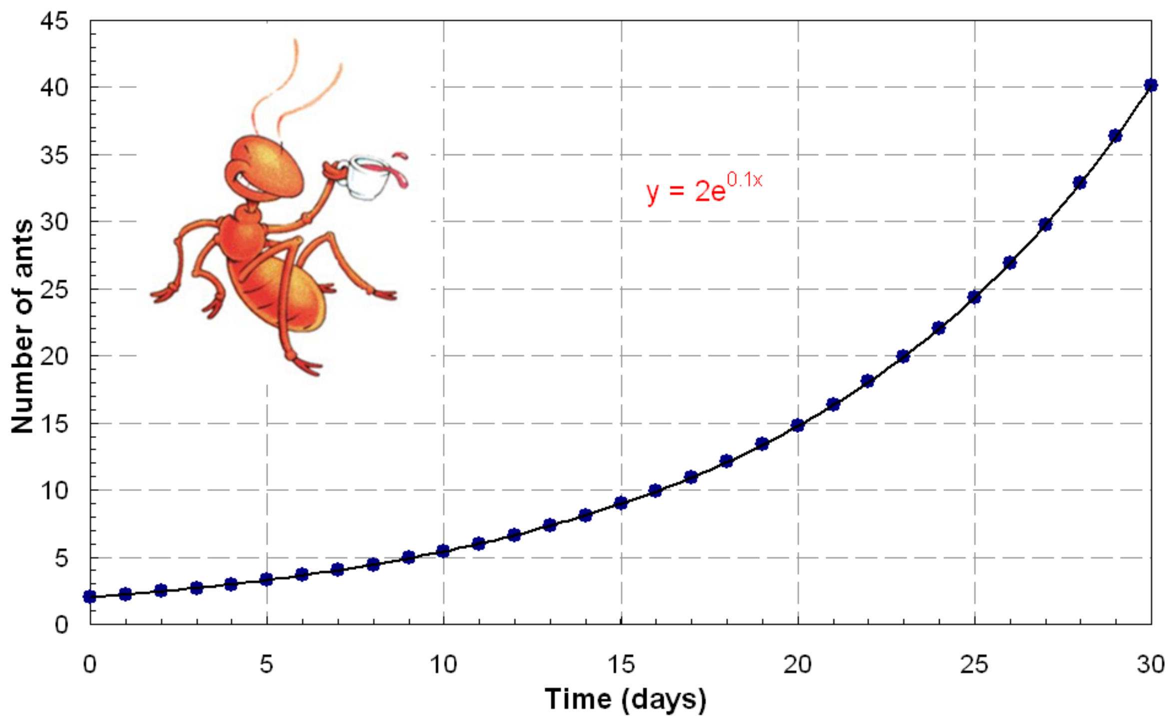
Growth rate $\frac{dP}{dt} = kP$ where P is the population

Solving gives $\ln P = kt + C$

$P = Ae^{kt}$ where $A = \ln C$

Condition $\begin{cases} t=0,\ P=2 & \Rightarrow A=2 \\ t=20,\ P=15 & \Rightarrow 15 = 2e^{20k} \Rightarrow k=0.1 \end{cases}$

So $P = 2e^{0.1t}$



3.2 Second Order ODEs

We will restrict our study of 2nd order ODEs to that of *linear equations with constant coefficients*

2nd order ODE have the form $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f(t)$.

We look first at equations with $f(t)=0$, called *homogeneous* or *unforced.* Next lecture we look at equations with $f(t)\neq 0$, called *inhomogeneous* or *forced* or *driven.* [*Note:* In this course we concentrate on the mathematics; the physics is further explored in PHY221.]

Homogeneous Equations - Revision of Theory

We have the equation $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$.

Looking for trial solutions of the form $x=e^{mt}$ leads to the or *auxiliary equation*

$$am^2+bm+c=0.$$

The roots of this equation are m1 and m2 and the general solution is

$$x=Ae^{m_1t}+Be^{m_2t}$$

- For real, distinct roots, m_1 and m_2, the general solution is $x=Ae^{m_1t}+Be^{m_2t}$
- For real, repeated roots, m, the general solution is $x=(At+B)e^{mt}$
- For complex roots $m=\alpha\pm i\beta$, the general solution may be written

$x=Ae^{(\alpha+i\beta)t}+Be^{(\alpha-i\beta)t}=e^{\alpha t}(Ae^{i\beta t}+Be^{-i\beta t})$ or equivalent form such as $x=e^{\alpha t}(C\sin\beta t+D\cos\beta t)=Ee^{\alpha t}[\cos(\beta t+\phi)]$.

NB. Proofs of these equivalent relationships can be found in the online problems.

Note that the general solution contains *two* arbitrary constants. *Two* boundary conditions must therefore be applied to find a particular solution.

Homogeneous Equations – Simplest examples with no damping or friction

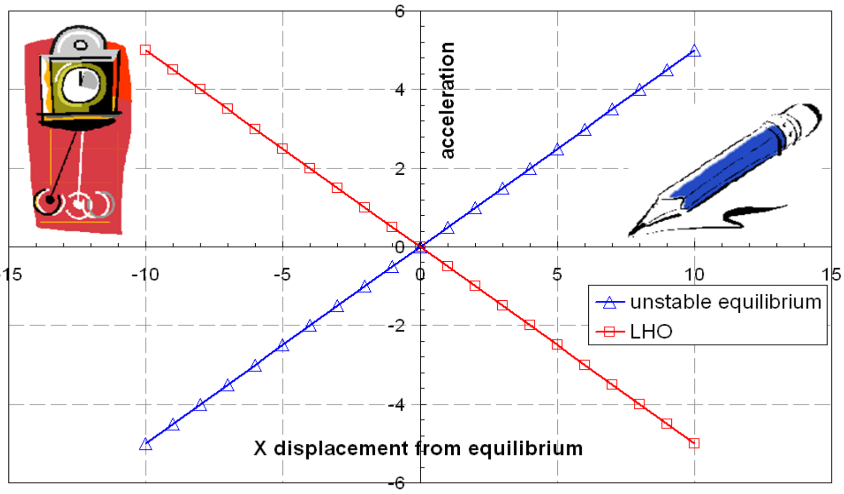
Two forms which occur very commonly in physics are:

1. Linear harmonic oscillator $\frac{d^2}{dt^2}x(t)=-\omega_0{}^2x(t)$ or $\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=0$

This equation occurs almost everywhere! E.g. all problems concerning waves (strings, light, etc.); small oscillations e.g. lattice vibrations in solids; LC electric circuits.

2. Unstable equilibrium $\frac{d^2}{dt^2}x(t)=\alpha^2x(t)$

This has less common occurrences as most systems in unstable equilibrium collapse… e.g. pencil balancing on its point.



Example 3.2 The Linear Harmonic Oscillator

Find the solution of $\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = 0$?

Substituting $x = e^{mt}$ yeilds auxilliary eq^n $m^2 + \omega_0^2 = 0$

Hence $m^2 = -\omega_0^2$, $m = \pm i\omega_0$

Complex roots means general sol^n is $x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

or $x(t) = C\sin\omega_0 t + D\cos\omega_0 t$

or $x(t) = E(\cos(\omega_0 t + \phi))$

Applying Boundary Conditions

If the particle starts at the origin with velocity V, i.e. $x(0) = 0$ and $\frac{dx(t)}{dt}\Big|_{t=0} = V$.

lets use the form $x(t) = C\sin\omega_0 t + D\cos\omega_0 t$

$x(0) = 0 = D + 0 \quad \Rightarrow D = 0$

$\frac{dx}{dt}\Big|_{t=0} = V \quad \Rightarrow V = C\omega_0 \cos(\omega_0 . 0) \quad \Rightarrow V = C\omega_0$

$\Rightarrow C = V/\omega_0$

Solution is $x(t) = \frac{V}{\omega_0}\sin\omega_0 t$

<u>Example 3.3</u> Unstable Equilibrium

Find the solution of $\frac{d^2}{dt^2}x(t) = \alpha^2 x(t)$?

$$\frac{d^2x}{dt^2} - \alpha^2 x = 0$$

Auxilliary eq^n: $m^2 - \alpha^2 = 0$

$m = \pm\alpha$

general sol^n is $x(t) = Ae^{\alpha t} + Be^{-\alpha t}$

Applying Boundary Conditions

Suppose $x(0) = L$ and $\frac{dx(t)}{dt}|_{t=0} = 0$. Apply the boundary conditions?

$x(t) = Ae^{\alpha t} + Be^{-\alpha t}$

at $t=0$, $x(t) = L$ So $L = A + B$

and $\left.\frac{dx}{dt}\right|_{t=0} = 0 \Rightarrow 0 = A\alpha - B\alpha$

$A = B$

So $A = B = \frac{L}{2}$ and the solution is $x(t) = \frac{L}{2}(e^{\alpha t} + e^{-\alpha t}) = L\cosh\alpha t$

Compare the solutions of equations (1) and (2). They have very different physical characteristics!

Solutions of (1) oscillate for ever.

Solutions of (2) grow to infinity as t increases.

3.3 Homogenous 2nd Order ODE's

INTRO: Hopefully these equations from PHY102 Waves & Quanta are familiar to you….

Free Oscillation with damping:

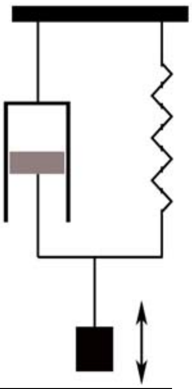
$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

Forced Oscillation with damping:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=H_0\cos\omega_D t$$

In this lecture we consider one more common homogeneous equation then two inhomogeneous equations.

Example 3.4 **The Damped Harmonic Oscillator** $\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$



Looking for solutions of the form e^{mt}
we obtain the characteristic equation $m^2+2\gamma m+\omega_0^2=0$.

This quadratic has two solutions: $m=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$

Be careful **!** There are **three** different cases.

(case i) $\gamma^2>\omega_0^2$ **(over-damping)**

We have two real values for *m*: $m_1=-\gamma+\sqrt{\gamma^2-\omega_0^2}$ and $m_2=-\gamma-\sqrt{\gamma^2-\omega_0^2}$.

And the general solution is $x(t)=Ae^{m_1t}+Be^{m_2t}$.

Both m_1 and m_2 are negative so $x(t)$ is the sum of two exponential decay terms and so tends pretty quickly, to zero. The effect of the spring has been made of secondary importance to the huge damping, e.g. fire doors.

(case ii) $\gamma^2=\omega_0^2$ **(critical damping)**

The characteristic equation has a double root $m=-\gamma$,
so the general solution is $x(t)=e^{-\omega t}[A+Bt]$ as shown earlier.
Here the damping has been reduced a little so the spring can act to change the displacement quicker. However the damping is still high enough that the displacement does not pass through the equilibrium position, e.g. car suspension – push down on the wheel arch and hope not to see SHM!

(case iii) $\gamma^2<\omega_0^2$ **(under-damping)**

The roots are complex. Define $\Omega^2=\omega_0^2-\gamma^2$ so $\sqrt{\omega_0^2-\gamma^2}=\pm\Omega$ and $\sqrt{\gamma^2-\omega_0^2}=\pm i\Omega$.
Then the two allowed values of m can be written $m_1=-\gamma+i\Omega$ and $m_2=-\gamma-i\Omega$.

The general solution can be written $x(t)=e^{-\gamma t}[Ae^{i\Omega t}+Be^{-i\Omega t}]$

or $x(t)=e^{-\gamma t}[C\cos\Omega t+D\sin\Omega t]$ or $x(t)=Fe^{-\gamma t}\cos(\Omega t+\phi)$.

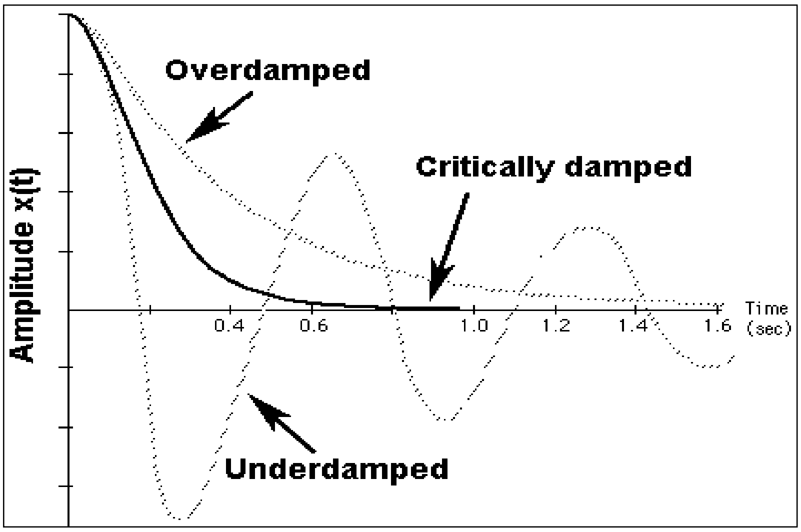
See the online Problems Lect3 Prob6.
The solution is the product of a sinusoidal term and an exponential decay term – so represents sinusoidal oscillations of decreasing amplitude. E.g. bed springs.

The amplitude will fall to $1/e$ of its original value after a time $\tau = \frac{1}{\gamma}$.

In many physically interesting cases $\gamma^2 << \omega_0^2$.
In this case $\Omega \sim \omega_0$, so $x(t) \approx Fe^{-\gamma t}\cos(\omega_0 t + \phi)$.

In that time τ the oscillator will have made n oscillations. $n = f\,\tau$ and $f = \frac{\omega_0}{2\pi}$ hence $n = \frac{\omega_0}{2\pi\gamma}$.

The ratio $\omega_0 / 2\gamma$ is called Q, the *quality factor*. Q is widely used in all areas of physics, a higher Q indicating a lower rate of energy dissipation relative to the oscillation frequency, so oscillations die more slowly. (see PHY102 topic 1 and PHY221).



3.4 Inhomogeneous 2nd order ODEs

We now look at *inhomogeneous* or *forced* second order linear ODEs with constant coefficients.

These are equations of the form $a\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+cx=f(t)$.

The two common driven equations which we will discuss are:

Example 3.5 & 3.6 $\dfrac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$ Driven oscillator no damping

Example 3.7 $\dfrac{d^2}{dt^2}x(t)+2\gamma\dfrac{dx(t)}{dt}+\omega_0^2x(t)=F\cos\omega t$ Damped driven oscillator

Equation 3.7 has applications in countless different areas of science! These include mechanical oscillators, LCR circuits, optics and lasers, NMR, nuclear physics, Mössbauer effect, pulsars, etc. etc. Equation 3.5 is usually unphysical, but it's much easier to solve, so we will look at this first!

Revision of Theory Solution involves four steps:

1) Find the general solution of the related homogeneous equation $a\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+cx=0$

   (by the methods discussed earlier). Call this **complementary solution** $x_c(t)$.

2) Find any solution of the full equation. This solution, $x_p(t)$, is often called a **particular solution**. It is found using an appropriate trial solution.

   e.g. If $f(t)=t^2$ try $x_p(t)=at^2+bt+c$
   If $f(t)=5e^{3t}$ try $x_p(t)=ae^{3t}$
   If $f(t)=5e^{i\omega t}$ try $x_p(t)=ae^{i\omega t}$
   If $f(t)=\sin 2t$ try $x_p(t)=a\cos 2t+b\sin 2t$
   (or complex version - see below)

   If $f(t)=\cos\omega t$ try $x_p(t)=Re[ae^{i\omega t}]$ see later for explanation
   If $f(t)=\sin\omega t$ try $x_p(t)=Im[ae^{i\omega t}]$

   If your trial solution has the correct form, substituting it into the differential equation will yield the values of the constants a, b, c, etc.

3) The complete general solution is the sum of the two parts above, $x=x_c+x_p$.

4) The complete general solution contains two constants (in x_c). If two boundary conditions are known, these should be applied to find the values of the constants.

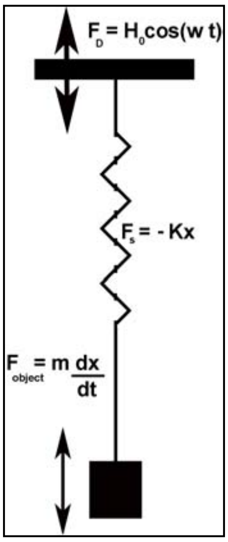
<u>Example 3.5</u> **The Undamped, Driven Oscillator** $\frac{d^2}{dt^2}x(t)+\omega_0^2 x(t)=F\cos\omega t$

Step 1 The corresponding homogeneous equation is simply the LHO equation. From the last lecture, therefore, we can take, say,

$$x_c(t)=A\cos\omega_0 t+B\sin\omega_0 t\ .$$

Step 2 We need to find the 'particular integral' using a trial solution. We should try

$$x_p(t) = a\cos\omega t + b\sin\omega t.$$



Substitute this trial solution into the original equation:

$$(\omega_0^2-\omega^2)a\cos\omega t + (\omega_0^2-\omega^2)b\sin\omega t = F\cos\omega t.$$

Comparing terms we can say that $b = 0$ and $(\omega_0^2-\omega^2)a = F$

Hence the trial solution is a solution provided

$$a=\frac{F}{\omega_0^2-\omega^2}\ ,\qquad \text{i.e.}\quad x_p(t)=\frac{F}{\omega_0^2-\omega^2}\cos\omega t\ .$$

Step 3 So the complete general solution is

$$x(t)=A\cos\omega_0 t+B\sin\omega_0 t+\frac{F}{\omega_0^2-\omega^2}\cos\omega t$$

Step 4 Suppose a particle subject to the equation above is known to be at rest at

$x=L$ at $t=0$.

This means we have the boundary conditions $x(0)=L$ and $\frac{dx}{dt}|_{t=0}=0$.

Substitute $t=0$ in the general solution given above:

$$x(0)=A+0+\frac{F}{\omega_0^2-\omega^2}=L$$

Differentiating the general solution, *then* substituting $t=0$ gives $\omega_0 B=0$

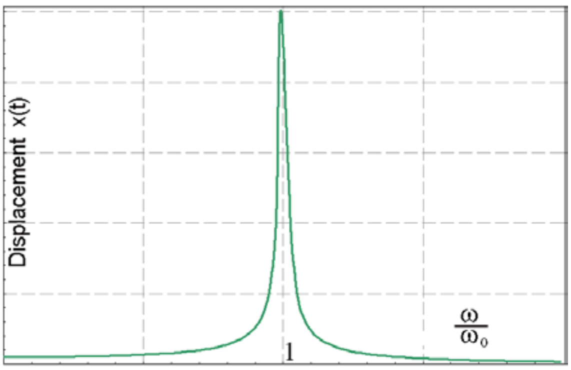
Hence $B=0$ and $A=L-\frac{F}{\omega_0^2-\omega^2}$ so the solution is:

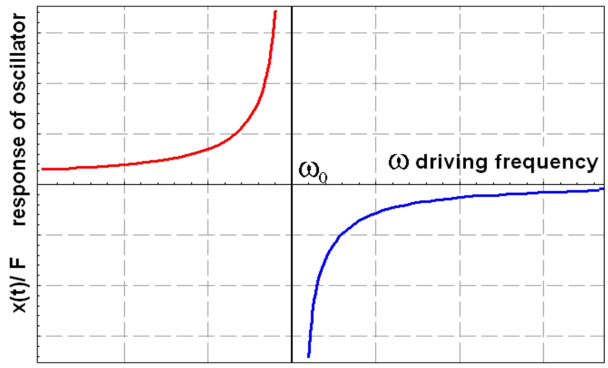
$$x(t)=(L-\frac{F}{\omega_0^2-\omega^2})\cos\omega_0 t+\frac{F}{\omega_0^2-\omega^2}\cos\omega t$$

This can be written as $x(t)=L\cos\omega_0 t+\frac{F}{(\omega_0^2-\omega^2)}(\cos\omega t-\cos\omega_0 t)$.

A few comments

1. Note that the solution is clearly not valid for $\omega = \omega_0$!
2. The ratio $\frac{x(t)}{F}$ is sometimes called the response of the oscillator. It is a function of ω. It is positive for $\omega < \omega_0$, negative for $\omega > \omega_0$. This means that at low frequency the oscillator follows the driving force but at high frequencies it is always going in the 'wrong' direction.





Example 3.6 Solution using Complex Numbers

The particular integral of the equation above was easy to find because a trial function of the form $x_p(t) = a\cos\omega t + b\sin\omega t$ worked. In our next equation (a driven oscillator with damping) this trial function would also work ... but the algebra gets very messy. It is easier to use complex numbers. To learn the complex method we will use it to solve equation 4 again for the particular integral.

Compare the original equation $\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = F\cos\omega t$ (A)

With the equation $\frac{d^2}{dt^2}X(t) + \omega_0^2 X(t) = Fe^{i\omega t}$ (B)

We know $F\cos\omega t = Re(Fe^{i\omega t})$, so *if equation (B) has (complex) solutions* $X(t)$ *then the solutions of equation (A) will be the real part of these:* $x(t) = Re(X(t))$. If the function on the RHS of (A) was $\sin\omega t$ then we could use the same approach but at the end take the imaginary part.

i.e. first we solve $\frac{d^2}{dt^2}X(t) + \omega_0^2 X(t) = Fe^{i\omega t}$.

This is easy: we take a trial solution of the form $X = Ae^{i\omega t}$.

Substituting this in gives: $(\frac{d^2}{dt^2} + \omega_0^2)Ae^{i\omega t} = (-\omega^2 + \omega_0^2)A(\omega)e^{i\omega t} = Fe^{i\omega t}$

Hence $A(\omega) = \dfrac{F}{(\omega_0^2 - \omega^2)}$ so $X(t) = \dfrac{F}{(\omega_0^2 - \omega^2)} e^{i\omega t}$

*To find the **particular solution** we* take the real part: $x(t) = Re(X(t)) = \dfrac{F \cos \omega t}{(\omega_0^2 - \omega^2)}$

<u>**Example 3.7**</u> **The Damped, Driven Oscillator** $\dfrac{d^2}{dt^2} x(t) + 2\gamma \dfrac{dx(t)}{dt} + \omega_0^2 x(t) = F \cos \omega t$

Step 1 The complementary function will be the solution of the damped harmonic oscillator, found at the beginning of this lecture. As discussed there, the appropriate form depends on the magnitude of γ compared to ω_0. However note that in every case, the solution tends to zero as
$t \to \infty$. It is often called the "transient" solution.

Step 2 The <u>particular integral</u>, by contrast, does not die away and is called the "steady state solution". We will find it using the complex method described above.

Consider the equation $\dfrac{d^2}{dt^2} X(t) + 2\gamma \dfrac{dX(t)}{dt} + \omega_0^2 X(t) = F \cos \omega t = F e^{i\omega t}$.

Look for solution of form $X = A(\omega) e^{i\omega t}$:

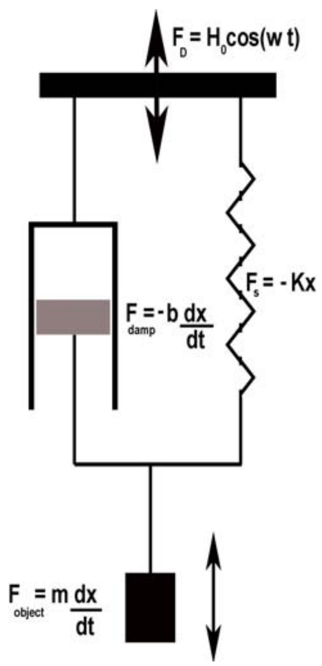
$$\left(\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2\right) A e^{i\omega t} = (-\omega^2 + 2i\omega\gamma + \omega_0^2) A e^{i\omega t} = F e^{i\omega t}$$

So $A(\omega) = \dfrac{F}{(-\omega^2 + 2i\omega\gamma + \omega_0^2)} = \dfrac{F}{Z(\omega)}$.

Remember to divide by a complex, we write it in form $e^{i\phi}$.

Let $(-\omega^2 + 2i\omega\gamma + \omega_0^2) = Z(\omega) = |Z(\omega)| e^{i\phi}$ where $|Z(\omega)| = \sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$ and $\tan \phi = \dfrac{2\gamma\omega}{\omega_0^2 - \omega^2}$.

Then $A(\omega) = \dfrac{F}{Z(\omega)} = \dfrac{F e^{-i\phi}}{|Z(\omega)|}$ so $X = \dfrac{F e^{-i\phi}}{|Z(\omega)|} e^{i\omega t} = \dfrac{F}{|Z(\omega)|} e^{i(\omega t - \phi)}$,



and now the <u>**last thing we do**</u> is to take the real part of the answer;

hence $x(t) = \mathrm{Re}[X] = \mathrm{Re}[\dfrac{F}{|Z(\omega)|} e^{i(\omega t - \phi)}]$

so $x(t) = \dfrac{F}{|Z(\omega)|} \cos(\omega t - \phi)$.

(**Steps 3 & 4** can then be followed if required.)

In cases where the damping is small, the amplitude has a strong peak at $\omega \approx \omega_0$ and the quality factor Q is again an important indicator.

Closing remarks

We have focussed on the *mathematics* of solving generic harmonic oscillator equations. By replacing ω, γ, etc. with appropriate constants, you should now be able to solve equations for all mechanical oscillators, oscillations in electrical LCR circuits, and numerous other oscillators! PHY221 and other courses will explore more of the physical significance of the solutions found here.

References

The material of lectures 3&4 is covered very thoroughly, with many real physical examples, by *French* in the course pack p.5-52:

Undamped, undriven LHO	7-9
Damped*, undriven LHO & Q-factor	10-16
Undamped, driven LHO: steady state	20-24
… again using complex exponentials	24-25
Damped*, driven LHO: steady state	25-28
Further discussion of Q, transients, resonance, etc.	31-42
Electrical, optical & nuclear examples	42-52

[*Note that French uses a damping constant γ while we have used 2γ]

Online problems (Topic 3, questions 1-11)

1. Verify the solution stated in the notes for $\frac{d^2x(t)}{dt^2} = \alpha^2 x(t)$ subject to $x(0) = L$ and $\left.\frac{dx(t)}{dt}\right|_{t=0} = 0$

2. Given that $\frac{d^2x(t)}{dt^2} = \alpha^2 x(t)$ and at $t = 0, x = 0$ and $v = \left.\frac{dx}{dt}\right|_{t=0}$ find solution and then plot x against αt.

3. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ with boundary conditions of $y(0) = 4$ and $\frac{dy(0)}{dx} = 0$

4. Solve $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 0$

5. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

6. Re-write your answer to 3 in terms of cos and sin, removing all complex formatting

7. Solve $\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 17y = 0$ with boundary conditions $y(0) = -4$ and $y'(0) = -1$

8. Atmospheric physics – a change in height Δh causes the pressure to drop by ΔP. This follows the equation $\Delta P = -\rho g \Delta h$ where ρ is the density of air. However the density is also a functions of the pressure P, so as the height increases the drop in pressure is not linear (as it would have been if p was constant). $\rho = \frac{mP}{kT}$ where m is the mass of one molecule, k is the Boltzmann constant and T is temperature and P is pressure. Write down the 1st order differential equation that defines the change in pressure with height and solve it.

9. The equation describing the process of discharging a capacitor which is initially charged to V_b is $R\frac{dQ}{dt} + \frac{Q}{C} = 0$ where $Q = CV_b$ at $t = 0$

10. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2\cos(3x)$

11. Solve $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} - 35y = \sin(2x)$

Topic 4. Fourier Series

References *Jordan & Smith* Ch.26, *Boas* Ch.7, *Kreyszig* Ch.11
Some fun 'java applet' demonstrations are available on the web. Try putting 'Fourier series applet' into Google and looking at the sites from jhu, Falstad and Maths Online Gallery.

4.1 Introduction to Fourier Series

Consider a length of string fixed between rigid supports. The full behaviour of this system can be found by solving a wave equation – a partial differential equation. We will do this later in the course. For now we will just recall the basic properties of waves of strings which we already know:

There is a ***fundamental*** mode of vibration.
Call the frequency of this mode f and the time period T.
Then there are various *harmonics*. These have frequency $2f, 3f, 4f, 5f, \ldots, nf, \ldots$

In practice, when a piano or guitar or other string is hit or plucked, it does not vibrate purely in one mode – the displacement of the string is not purely sinusoidal, the sound emitted is not all of one frequency. In practice, one normally hears a large amount of the fundamental *plus* smaller amounts of various harmonics. The proportions in which the different frequencies are present varies – hence a guitar sounds different from a violin or a piano, and a violin sounds different if it is bowed from if it is plucked!

(See http://www.jhu.edu/~signals/listen/music1.html pages 1&2)

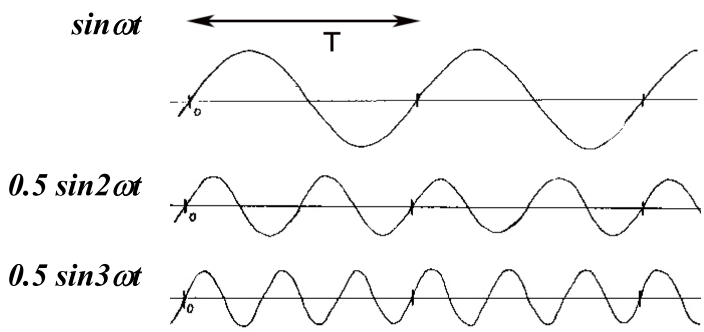
Remember that if the fundamental frequency has frequency f, its period $T = 1/f$.

A harmonic wave of frequency nf will then have a period T/n, but obviously also repeats with period T.

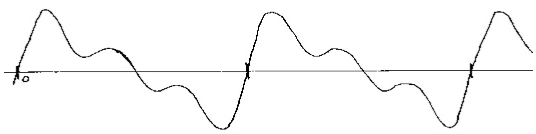
So if we add together sinusoidal waves of frequency $f, 2f, 3f, 4f, \ldots$ the result is a (non-sinusoidal) waveform which is periodic with the same period T as the fundamental frequency, $f = 1/T$.

[E.g. play with http://www.falstad.com/fourier/]

Sometimes we use the angular frequency ω where the nth harmonic has $\omega_n = 2\pi nf = 2\pi n/T$.
The various harmonics are then of the form $A\sin\omega_n t$.



$y_1(t) = \sin\omega t + 0.5\sin 2\omega t + 0.5\sin 3\omega t$



For all the functions above, the *average value over a period* is zero. If we add a *constant* term, the waveform remains periodic but its average value is no longer zero:

$y_2(t) = 1 + \sin\omega t + 0.5\sin 2\omega t + 0.5\sin 3\omega t$

What is really *useful* is that this works in reverse:

<u>Any</u> periodic function with period T can be expressed as the sum of a constant term plus harmonic (sine and cosine) curves of angular frequency $\omega, 2\omega, 3\omega, \ldots$ where $\omega = 2\pi/T$.

i.e. we can write

$$\begin{aligned} F(t) &= \frac{1}{2}a_0 + (a_1\cos\omega t + b_1\sin\omega t) + (a_2\cos 2\omega t + b_2\sin 2\omega t) + \ldots \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega t + b_n\sin n\omega t \end{aligned}$$

where $\omega = 2\pi/T$.

We will later prove this result mathematically, and later in the semester will see that it can be deduced from the general solution of the wave equation. For now you may be able to persuade yourself of its plausibility by playing with the various websites – for example, the demonstrations of how 'square' or 'triangular' waveforms can be made from sums of harmonic waves. The more terms in the sum, the closer the approximation to the desired waveform. Hence in general, an infinite number of terms are needed.

4.2 Why is this useful?

In lecture 4 we solved the 'forced harmonic oscillator' equation

$$\frac{d^2}{dt^2}x(t) + 2\gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = F\cos\omega t\ .$$

Such an equation could describe, for example, the response of an electrical LCR circuit to a sinusoidal driving voltage. But what would happen if we applied a *square wave* driving voltage?? Using Fourier theory, we would just need to express the square waveform as a sum of sinusoidal terms. Then the response would be the sum of the solutions for each term (which would all have similar form, but involve different multiples of ω thus also have different amplitudes). Throughout physics there are many similar situations. Fourier series means that *if we can solve a problem for a sinusoidal function then we can solve it for <u>any</u> periodic function*!

And periodic functions appear everywhere! Examples of periodicity in time: a pulsar, a train of electrical pulses, the temperature variation over 24 hours or the average daily temperature over a year (approximately). Examples of periodicity in space: a crystal lattice, an array of magnetic domains, etc.

Other Forms

If we want to work in terms of t not ω, the formula becomes

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}.$$

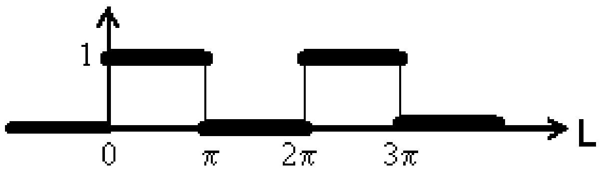
Or similarly for a function $f(x)$ which is periodic in *space* with repetition length L, we have

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}.$$

(Any value of T or L can be used, although to keep the algebra straight forward, most questions will set T as 2π or even L as 2π metres.)

4.3 Towards Finding the Fourier Coefficients

To make things easy let's say that the pattern repeats itself every 2π metres, so $L = 2\pi$.



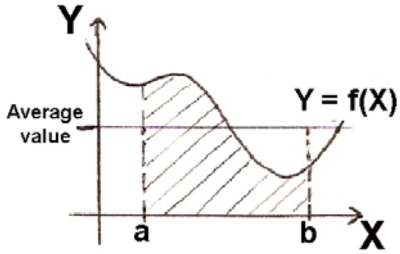
The Fourier series can then be expressed more simply in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Now we want to find expressions for the coefficients a_n and b_n.

To do this we need two other bits of preparatory mathematics …

4.4 Average Value of a Function



Consider a function $y = f(x)$. The *average value* of the function between $x = a$ and $x = b$ is defined to be $\frac{1}{b-a}\int_a^b f(x)dx$.

Geometrically this means that the area under the curve $f(x)$ between a and b is equal to the area of a rectangle of width $(b-a)$ and height equal to this average value. Note that while average values can be found by evaluating the above integral, sometimes they can be identified more quickly from symmetry considerations, a sketch graph and common sense!

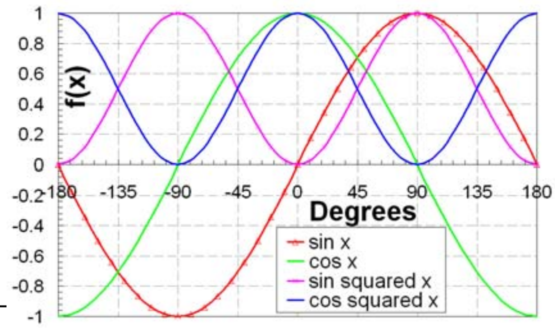
Two particularly important results are:

The average value of a sine or cosine function over a period is zero:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos nx\,dx = 0.$$

The average value of $\cos^2$ *or* $\sin^2$ *over a period is* ½:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,dx = \frac{1}{2}.$$



Actually both these results can be generalized. It is easily shown that:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos nx\,dx = 0 \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi}\sin^2 nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos^2 nx\,dx = \frac{1}{2} \quad \text{for } n \neq 0$$

Hence $\int_0^{2\pi}\sin nx\,dx = \int_0^{2\pi}\cos nx\,dx = 0$ and $\int_0^{2\pi}\sin^2 nx\,dx = \int_0^{2\pi}\cos^2 nx\,dx = \pi \quad (n \neq 0)$

Note: **We have written all the integrals over $[0, 2\pi]$ but *any* interval of width 2π can be used**, e.g. $[-\pi, \pi]$, $[13.1\pi, 15.1\pi]$, etc.

4.5 Orthogonality (Proofs in the Appendix)

Sines and cosines have an important property called 'orthogonality':
The product of two different sine or cosine functions, integrated over a period, gives zero:

$$\int_0^{2\pi}\sin nx\cos mx\,dx = 0 \quad \text{for all } n, m$$

$$\int_0^{2\pi}\sin nx\sin mx\,dx = \int_0^{2\pi}\cos nx\cos mx\,dx = 0$$

for all $n \neq m$

Again we can integrate over *any period.* Equipped with these results we can now find the Fourier coefficients …

4.6 Fourier Coefficients – Derivation

Earlier we said any function $f(x)$ with period 2π can be written

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx .$$

Take this equation and integrate both sides over a period (any period):

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0 \int_0^{2\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx\, dx + b_n \int_0^{2\pi} \sin nx\, dx \right]$$

Clearly on the RHS the only non-zero term is the a_0 term:

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0 \int_0^{2\pi} dx = \frac{1}{2}a_0(2\pi - 0) = \pi a_0$$

hence $\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx}$ i.e. $a_0/2$ is the *average value of the function* $f(x)$.

Now take the original equation again, multiply both sides by $\cos x$, then integrate over a period:

$$\int_0^{2\pi} f(x)\cos x dx = \frac{1}{2}a_0 \int_0^{2\pi} \cos x dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx \cos x\, dx + b_n \int_0^{2\pi} \sin nx \cos x\, dx \right]$$

On the RHS, this time only the a_1 term survives as it is the only term where $n=1$ (see Orthogonality.)

$$\int_0^{2\pi} f(x)\cos x dx = a_1 \int_0^{2\pi} \cos x \cos x\, dx = a_1 \int_0^{2\pi} \cos^2 x\, dx = a_1 \pi$$

hence $\boxed{a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx}$.

The method for finding the coefficients a_n should thus be clear. To find a general expression for a_n we can take the equation, multiply both sides by $\cos mx$, then integrate over a period:

$$\int_0^{2\pi} f(x)\cos mx dx = \frac{1}{2}a_0 \int_0^{2\pi} \cos mx dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx \cos mx\, dx + b_n \int_0^{2\pi} \sin nx \cos mx\, dx \right]$$

On the RHS, only the a_m term survives the integration:

$$\int_0^{2\pi} f(x)\cos mx dx = a_m \int_0^{2\pi} \cos^2 mx\, dx = a_m \pi$$ hence $\boxed{a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx dx}$.

In a similar way, multiplying both sides by $\sin mx$, then integrating over a period we get:

$$\boxed{b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\, dx}$$

4.7 Summary of Results

A function $f(x)$ with **period** 2π can be expressed as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$.

The more general expression from page 2 can be written as:-

A function $f(x)$ with **period** L can be expressed as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

where $a_0 = \frac{2}{L}\int_0^{L} f(x)dx$, $a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{2n\pi x}{L} dx$, $b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{2n\pi x}{L} dx$.

<u>Note</u>

1) The formula for a_0 can be obtained from the formula for a_n just by setting $n = 0$.

2) The integrals above are written over $[0, 2\pi]$ and $[0, L]$ but *any* convenient interval of width one period may be used, and this will be dependent on the nature of the function (see examples and the online Problems).

3) The equations can be easily adapted to work with other variables or periodicities. For example, for a function periodic in time with period T just replace x by t and L by T.

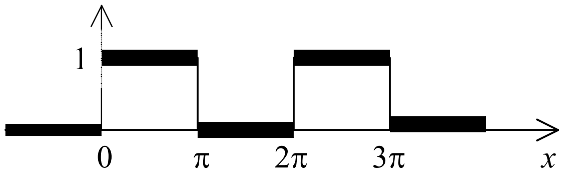
4) A few books use the alternative form

$$F(t) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t + \theta_n)$$ and find values of d_n and θ_n.

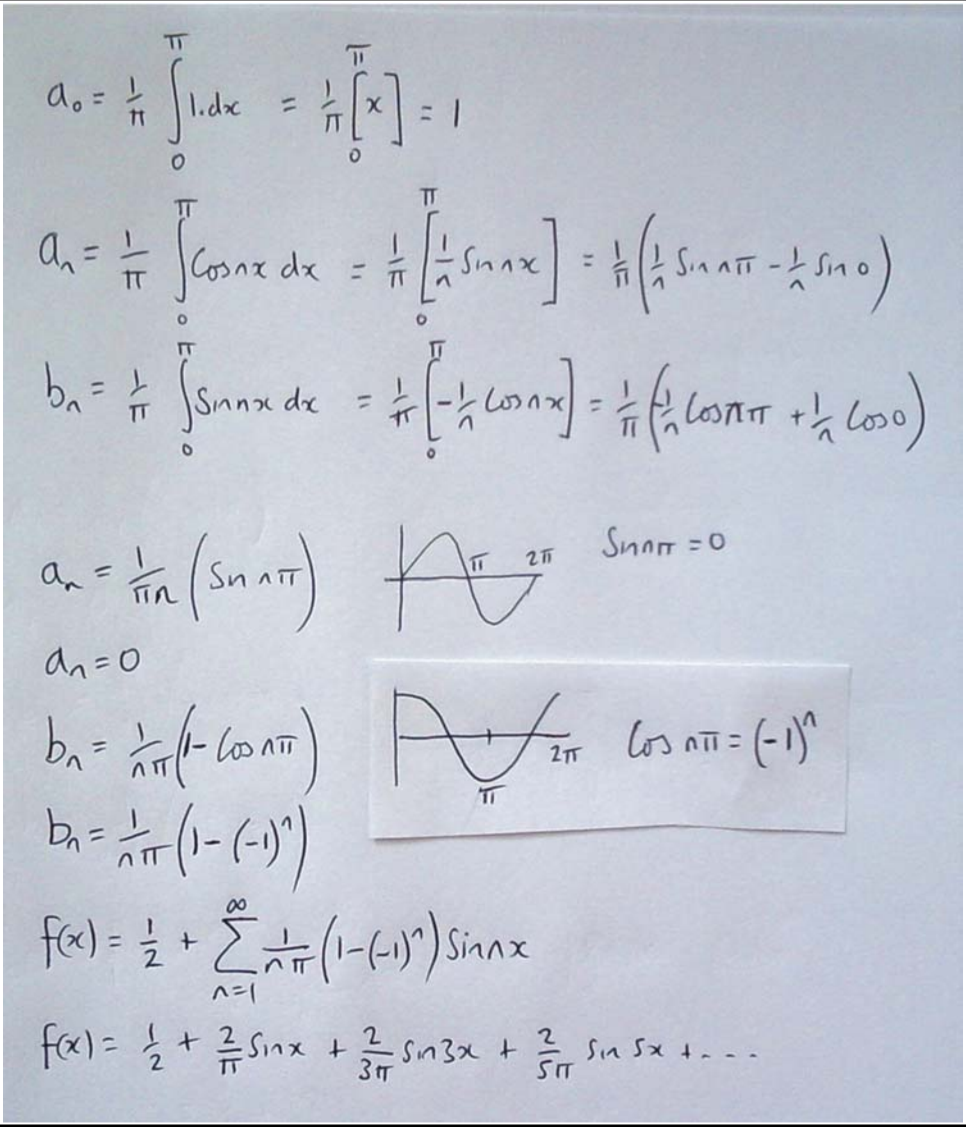
4.8 Examples

Example 4.1

Find a Fourier series for the square wave shown.



We have $f(x)=\begin{cases}1 & 0<x<\pi \\ 0 & \pi<x<2\pi\end{cases}$ The period is 2π.



$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1.dx = \frac{1}{\pi}\Big[x\Big]_0^{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos nx\, dx = \frac{1}{\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{n}\sin n\pi - \frac{1}{n}\sin 0\right)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{1}{n}\cos n\pi + \frac{1}{n}\cos 0\right)$$

$$a_n = \frac{1}{\pi n}\left(\sin n\pi\right) \qquad \sin n\pi = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{n\pi}\left(1-\cos n\pi\right) \qquad \cos n\pi = (-1)^n$$

$$b_n = \frac{1}{n\pi}\left(1-(-1)^n\right)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left(1-(-1)^n\right)\sin nx$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \dots$$

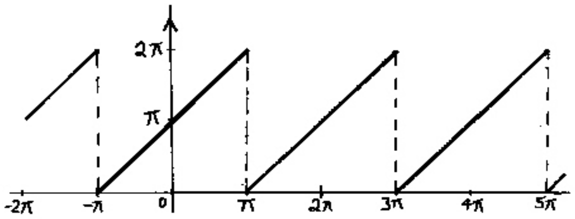
Think: After reaching your answer, ask yourself: *is this result sensible?*

- Does the term $a_0/2$ look like an appropriate value for the average value of the function over a period?
- Would we expect this function to be made mainly of sines or of cosines? (See later for symmetry).
- In what proportions would we expect to find the fundamental and the various harmonics?

(You can also try checking your answer by 'building' the series at http://www.falstad.com/fourier/ or http://www.univie.ac.at/future.media/moe/galerie/fourier/fourier.html)

Example 4.2

Find a Fourier series of the function shown:



Again the period is 2π.
But this time it is easiest to work with the range $[-\pi, \pi]$. If we wanted we could use the range $[0,2\pi]$ and get the same answer, but it would be fiddlier.

Between $-\pi$ and π, $f(x)$ is a straight line with gradient 1 and a Y-intercept of π.
So we can write $f(x) = x + \pi \quad -\pi < x < \pi.$

Lets find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\pi x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\left\{\frac{\pi^2}{2}+\pi^2\right\}-\left\{\frac{\pi^2}{2}-\pi^2\right\}\right) = \frac{1}{\pi}\left(2\pi^2\right) = 2\pi$$

Lets find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\cos nx\,dx$$

We must integrate $\int_{-\pi}^{\pi} x\cos nx\,dx$ by parts:

$$\int u\,dv = uv - \int v\,du$$

$u = x$
$dv = \cos nx\, dx$

$du = dx$
$v = \int \cos nx dx = \frac{1}{n}\sin nx$

So $\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{x}{n}\sin nx - \int_{-\pi}^{\pi}\frac{1}{n}\sin nx dx = \left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} = 0$
(see p.3 average value)

Going back to a_n,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\cos nx\,dx = 0$$
(see p.3 average value)

Now let's find the b_n coefficients....

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\sin nx\,dx$$

We must integrate $\int_{-\pi}^{\pi} x\sin nx\,dx$ by parts:

$$\int u\,dv = uv - \int v\,du$$

u $= x$
$dv = \sin nx\; dx$

$du = dx$
$v = \int \sin nx dx = -\frac{1}{n}\cos nx$

So $\int_{-\pi}^{\pi} x\sin nx\,dx = -\frac{x}{n}\cos nx - \int_{-\pi}^{\pi}\frac{-1}{n}\cos nx dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$

(see p.3 average value)

Going back to b_n,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\sin nx\,dx = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} + 0$$

$$b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\left(\frac{-\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi\right) - \left(\frac{-(-\pi)}{n}\cos(-n\pi) + \frac{1}{n^2}\sin(-n\pi)\right)\right]$$

Remember that $\cos(-n\pi) = \cos(n\pi)$ and $\sin(-n\pi) = -\sin(n\pi)$

So $b_n = \frac{1}{\pi}\left[\left(\frac{-2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi\right)\right] = -\frac{2}{n}\cos n\pi$ What will b_n be for different values of n?

n = 1	n = 2	n = 3	n = 4
$-\frac{2}{1}\cos 1\pi = -\frac{2}{1}(-1) = \frac{2}{1}$	$-\frac{2}{2}\cos 2\pi = -\frac{2}{2}(1) = -\frac{2}{2}$	$-\frac{2}{3}\cos 3\pi = -\frac{2}{3}(-1) = \frac{2}{3}$	$-\frac{2}{4}\cos 4\pi = -\frac{2}{4}(1) = -\frac{2}{4}$

Hence $f(x) = \frac{2\pi}{2} + 0 + \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + ... = \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

Notes

1) Where a function has discontinuities, the Fourier Series converges to the midpoint of the jump (e.g. in example 1 at $x = 0, \pi$, etc the series has value ½).
2) In general the lowest frequency terms provide the main shape, the higher harmonics add the detail. When functions have discontinuities, more higher harmonics are needed. Hence in both the above examples the terms drop off quite slowly. In general, for smoother functions the terms drop off faster.

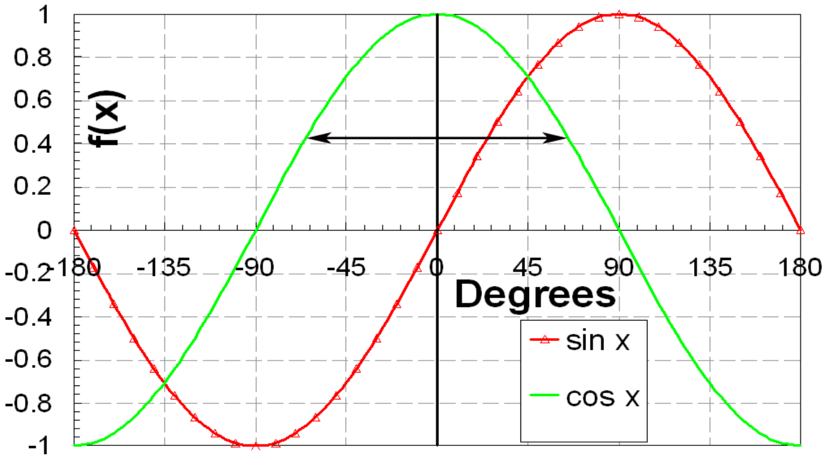
4.9 Even and Odd Functions

For an *even* function, $f_e(-x) = f_e(x)$

i.e. the graph $y = f(x)$ has reflection symmetry in the y-axis.

For an *odd* function, $f_o(-x) = -f_o(x)$

i.e. the graph $y = f(x)$ has 180º rotational symmetry about the origin.



Any sum of even functions is also an even function. Hence $\sum_{n=0}^{\infty} a_n \cos nx$ is always an even function.

Therefore **the Fourier series of an even function contains only cosine terms**.

Similarly, **the Fourier series of an odd function contains only sine terms**.

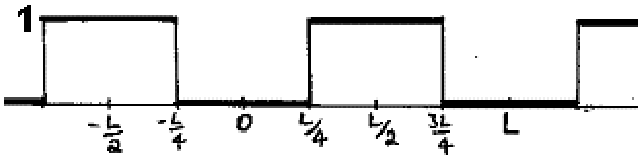
It is **exceptionally useful** to remember this! E.g. if you are asked to find the Fourier series of a function which is even, you can immediately state that $b_n = 0$ for all n, meaning that there will be no sine terms.

You should also remember the following facts (easily verified algebraically or by sketching graphs):

- The product of an *even* function and an *even* function is *even*
- The product of an *odd* function and an *odd* function is *even*
- The product of an *even* function and an *odd* function is *odd*

Example 4.3

Find a Fourier series of the function shown:



The period is L. As discussed earlier we can integrate over any full period e.g. $\int_0^L$ or $\int_{-L/2}^{L/2}$

The function is *even* and can be written $f(x) = 1$ for $L/4 \le x \le 3L/4$. Therefore there will be no sine terms ($b_n = 0$ for all n) and I feel like integrating between 0 and L. The series will have form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} \quad \text{where} \quad a_0 = \frac{2}{L}\int_0^L f(x)dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx.$$

So $a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{L}\int_{L/4}^{3L/4} 1dx = \frac{2}{L}[x]_{L/4}^{3L/4} = 1$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx = \frac{2}{L}\int_{L/4}^{3L/4} 1\cos\frac{2n\pi x}{L}dx = \frac{2L}{2n\pi L}\left[\sin\frac{2n\pi x}{L}\right]_{L/4}^{3L/4} = \frac{1}{n\pi}\left\{(\sin\frac{6n\pi}{4}) - (\sin\frac{2n\pi}{4})\right\}$$

$$a_n = \frac{1}{n\pi}\left\{(\sin\frac{6n\pi}{4}) - (\sin\frac{2n\pi}{4})\right\} = \frac{1}{n\pi}\left\{(\sin\frac{3n\pi}{2}) - (\sin\frac{n\pi}{2})\right\}$$

Expression for a_n is not very pretty and easy to make mistakes with. Write out a table to help with assignment of coefficients….

n = 1	n = 2	n = 3	n = 4
$\frac{1}{\pi}\left\{(\sin\frac{3\pi}{2}) - (\sin\frac{\pi}{2})\right\} = -\frac{2}{\pi}$	$\frac{1}{2\pi}\left\{(\sin\frac{6\pi}{2}) - (\sin\frac{2\pi}{2})\right\} = 0$	$\frac{1}{3\pi}\left\{(\sin\frac{9\pi}{2}) - (\sin\frac{3\pi}{2})\right\} = \frac{2}{3\pi}$	$\frac{1}{4\pi}\left\{(\sin\frac{12\pi}{2}) - (\sin\frac{4\pi}{2})\right\} = 0$
n = 5	**n = 6**	**n = 7**	**n = 8**
$\frac{1}{5\pi}\left\{(\sin\frac{15\pi}{2}) - (\sin\frac{5\pi}{2})\right\} = -\frac{2}{5\pi}$	$\frac{1}{6\pi}\left\{(\sin\frac{18\pi}{2}) - (\sin\frac{6\pi}{2})\right\} = 0$	$\frac{1}{7\pi}\left\{(\sin\frac{21\pi}{2}) - (\sin\frac{7\pi}{2})\right\} = \frac{2}{7\pi}$	$\frac{1}{8\pi}\left\{(\sin\frac{24\pi}{2}) - (\sin\frac{8\pi}{2})\right\} = 0$

So $f(x) = \frac{1}{2} - \left(\frac{2}{\pi}\cos\frac{2\pi x}{L}\right) + \left(\frac{2}{3\pi}\cos\frac{6\pi x}{L}\right) - \left(\frac{2}{5\pi}\cos\frac{10\pi x}{L}\right) + \ldots$

4.10 Half-Range Series

Sometimes we want to find a Fourier series representation of a function which is valid just over some restricted interval. We could do this in the normal way and then state that the function is only valid over a specific interval. However, the fact that we can do this allows us to use a clever trick that reduces the complexity of a problem. We will study this by considering the following example:

Example 4

Consider a guitar string of length L that is being plucked.

(*Note on application*: If a string was released from this position, finding this Fourier series would be a crucial step in determining the displacement of the string at all subsequent times – see later in course.)

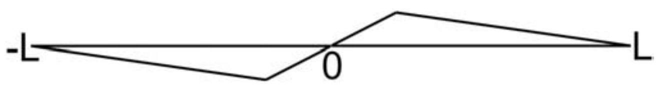
We could, as before, apply the Fourier series to a pretend infinite series of plucked strings and then say that the expression was only valid between 0 and L.

However this series would contain both sine and cosine terms as there is neither even nor odd symmetry, and so would take ages to solve. There is a much more clever way to proceed….

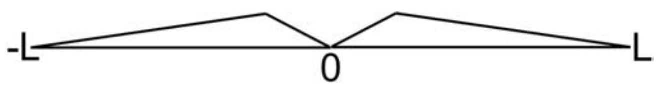
Note that we are only told the form of the function on the interval $[0, L]$. All that matters is that the series corresponds to the given function *in the given interval*. What happens outside the given interval is irrelevant. The way to tackle such a problem is to consider an *artificial function* which coincides with the given function over the given interval… **<u>but extends it and is periodic</u>**.

Clearly we could do this in an infinite number of different ways, however in the previous section, we observed that the Fourier series of odd and even functions are particularly simple. It is therefore sensible to choose an odd or even artificial function!

If the original function is defined on the range $[0, L]$ then there are always odd and even artificial functions with period $2L$. In this case these look like



Odd extension



Even extension

These functions are called the **odd extension** and **even extension** respectively.
Their corresponding Fourier series are called the **half-range sine series** and **half-range cosine series**.

Theory

We saw earlier that for a function with period *L* the Fourier series is:-

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L},\ \text{where}\ a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx,$$
$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

In this case we have a function of period $2L$ so the formulae become

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x,\ \text{where}\ a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$
$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

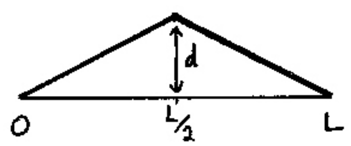
Remembering also that $\int_{-b}^{b}f_e(x)dx=2\int_0^b f_e(x)dx$, we get the following results:

Half-range cosine series:	$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$,	where $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$.
Half-range sine series:	$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$,	where $b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$.

Note : The resulting series is only valid over the specified interval!

Example 4.4

Find a Fourier series which represents the displacement $y(x)$, between $x = 0$ and L, of the 'plucked string' shown.



Let us choose to find the half-range *sine* series.

We have $y(x)=\begin{cases}2x\,d/L & 0<x<L/2\\ 2(L-x)d/L & L/2<x<L\end{cases}$

So $b_m=\frac{2}{L}\int_0^L dx\sin\frac{m\pi x}{L}Y(x)=\frac{2}{L}\int_0^{L/2}dx\frac{2dx}{L}\sin\frac{m\pi x}{L}+\frac{2}{L}\int_{L/2}^{L}dx\frac{2d}{L}(L-x)\sin\frac{m\pi x}{L}$

Using integration by parts, it can be shown that the result is:

$$b_n\ =\frac{8d}{\pi^2m^2}\sin\frac{m\pi}{2}\qquad \textit{for}\quad m\quad \textit{odd}$$

$$b_n\ =\ 0\qquad \textit{for}\quad m\quad \textit{even}$$

So for $0<x<L$ we have $Y(x)=\frac{8d}{\pi^2}\left[\sin\frac{\pi x}{L}-\frac{1}{9}\sin\frac{3\pi x}{L}+\frac{1}{25}\sin\frac{5\pi x}{L}-\frac{1}{49}\sin\frac{7\pi x}{L}+.........\right]$

Work out the full solution for yourself. This question is answered in the online problems.

Further Results

4.11 Complex Series.

For the waves on strings we need real standing waves. But in some other areas of physics, especially solid state physics, it is more convenient to consider complex or running waves. Remember that:

$$\cos kx = \frac{1}{2}(e^{ikx} + e^{-ikx}); \qquad \sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx}) = \frac{-i}{2}(e^{ikx} - e^{-ikx})$$

The complex form of the Fourier series can be derived by assuming a solution of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and then by evaluating the coefficients as in section 3, taking the expression and multiplying both sides by e^{-imx} and integrating over a period:

$$\int_0^{2\pi} f(x)e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{inx}e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-m)x}dx$$

For $n \neq m$ the integral vanishes. For $n=m$ the integral gives 2π. Hence

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

Complex Fourier Series for a function of period 2π:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

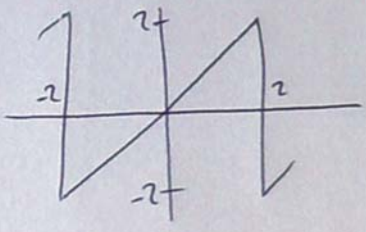
A function $f(x)$ with **period *L*** can be expressed as:-

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L} \quad \text{where} \quad c_n = \frac{1}{L}\int_0^{L} f(x)e^{-2\pi inx/L}dx$$

Let's have a look at an example of complex Fourier series.

Example 4.5

Find the complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$ if the repeat period is 4.



$$f(x) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n x/L}$$

$$c_n = \frac{1}{L}\int_0^L f(x)\, e^{-2\pi i n x/L}\, dx$$

Period of 4 from -2 to 2

$$c_n = \frac{1}{4}\int_{-2}^{2} x\, e^{-\pi i n x/2}\, dx$$

Integrate by parts

$$\int u\,dv = uv - \int v\,du$$

$u = x$

$du = dx$

$dv = e^{-\pi i n x/2}\,dx$

$V = \frac{-2}{\pi i n} e^{-\pi i n x/2}$

$$c_n = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{-\pi i n x/2} + \frac{2}{\pi i n}\int e^{-\pi i n x/2}\,dx\right]_{-2}^{2}$$

$$c_n = \left[\frac{-x}{2\pi i n} e^{-\pi i n x/2} - \frac{1}{\pi^2 i^2 n^2} e^{-\pi i n x/2}\right]_{-2}^{2}$$

$$c_n = \left[\frac{-x\, e^{-\pi i n x/2}}{2\pi i n} + \frac{1}{\pi^2 n^2} e^{-\pi i n x/2}\right]_{-2}^{2}$$

Since $\frac{1}{i^2} = \frac{1}{-1} = -1$

$$c_n = \left(\frac{-2e^{-\pi i n}}{2\pi i n} + \frac{1}{\pi^2 n^2} e^{-\pi i n}\right) - \left(\frac{2e^{\pi i n}}{2\pi i n} + \frac{1}{\pi^2 n^2} e^{\pi i n}\right)$$

Remember $\frac{1}{i} = -i$

$$c_n = \frac{i}{\pi n}\left(e^{-\pi i n} + e^{\pi i n}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi i n} - e^{\pi i n}\right)$$

$e^{-\pi i n} + e^{\pi i n} = 2\cos \pi n = 2(-1)^n$

$$c_n = \frac{2i}{\pi n}(-1)^n$$

$e^{-\pi i n} - e^{\pi i n} = -2i \sin \pi n = 0$

So $$f(x) = \sum_{-\infty}^{\infty} \frac{2i}{\pi n}(-1)^n e^{i\pi x/2}$$

4.12 Parseval's Theorem

Consider again the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \, .$$

Square both sides then integrate over a period:

$$\int_0^{2\pi} [f(x)]^2 dx = \int_0^{2\pi} \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right]^2 dx$$

The RHS will give both squared terms and cross term. When we integrate, *all* the cross terms will vanish. All the squares of the cosines and sines integrate to give π (half the period. Hence…

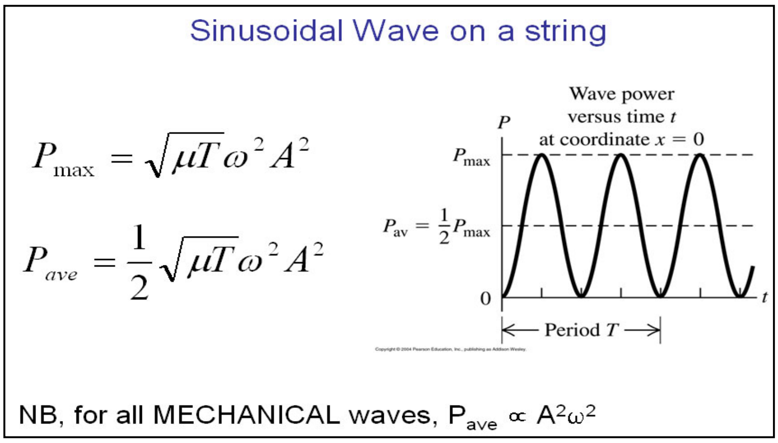
$$\int_0^{2\pi} [f(x)]^2 dx = 2\pi \frac{a_0{}^2}{4} + \pi \sum_{n=1}^{\infty} [a_n{}^2 + b_n{}^2]$$

The energy in a vibrating string or an electrical signal is proportional to an integral like

$$\int_0^{2\pi} [f(x)]^2 dx \, .$$

Hence Parseval's theorem tells us that the total energy in a vibrating system is equal to the sum of the energies in the individual modes.

From PHY101…



4.13 Appendix: Orthogonality

At a fundamental mathematical level, the reason the Fourier series works – the reason any periodic function can be expressed as a sum of sine and cosine functions – is that sines and cosines are orthogonal.

In general, a set of functions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ is said to be *orthogonal* on the interval $[a, b]$ if

$$\int_a^b u_n(x)u_m(x)\,dx = \begin{cases} 0 & n \neq m \\ c_n & n = m \end{cases}$$ (where c_n is a constant).

Here we will prove that function $\sin nx$, $\cos mx$, etc are orthogonal on the interval $[0, 2\pi]$.

1. $\int_0^{2\pi} \sin nx \cos mx\,dx = \frac{1}{2}\int_0^{2\pi} \sin(n+m)x - \sin(n-m)x\,dx$

[Using $\sin(a+b) - \sin(a-b) = 2\sin a \cos b$]

$$= \frac{1}{2}\left[-\frac{1}{n+m}\cos(n+m)x + \frac{1}{n-m}\cos(n-m)x\right]_0^{2\pi} = 0$$

Hence $\int_0^{2\pi} \sin nx \cos mx\,dx = 0$ for $n \neq m$

2. $\int_0^{2\pi} \sin nx \sin mx\,dx = \frac{1}{2}\int_0^{2\pi} \cos(n-m)x - \cos(n+m)x\,dx$

[Using $\cos(a-b) - \cos(a+b) = 2\sin a \sin b$]

$$= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n-m}\sin(n-m)x\right]_0^{2\pi} = 0$$

Hence $\int_0^{2\pi} \sin nx \sin mx\,dx = 0$ for $n \neq m$.

3. $\int_0^{2\pi} \cos nx \cos mx\,dx = \frac{1}{2}\int_0^{2\pi} \cos(n+m)x + \cos(n-m)x\,dx$

[Using $\cos(a-b) + \cos(a+b) = 2\cos a \cos b$]

$$= \frac{1}{2}\left[\frac{1}{n+m}\sin(n+m)x + \frac{1}{n-m}\sin(n-m)x\right]_0^{2\pi} = 0$$

Hence $\int_0^{2\pi} \cos nx \cos mx\,dx = 0$ for $n \neq m$

For $n = m \neq 0$ the integrals becomes:

1. $\int_0^{2\pi} \sin nx \cos nx \, dx = \frac{1}{2}\int_0^{2\pi} \sin 2nx \, dx = \left[-\frac{1}{4n}\cos 2nx \right]_0^{2\pi} = 0$

2. $\int_0^{2\pi} \sin^2 nx \, dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2nx) \, dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx \right]_0^{2\pi} = \pi$

3. $\int_0^{2\pi} \cos^2 nx \, dx = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2nx) \, dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin 2nx \right]_0^{2\pi} = \pi$

For $n = m = 0$ the first two integrals become $\int_0^{2\pi} 0 dx = 0$ and the third becomes $\int_0^{2\pi} 1 dx = 2\pi$

Note

1. Similar results can be proved for function of periodicity *L.*
2. The results $(n \neq 0)$ are easy to remember: ALL integrals over sines and cosines over a full period give zero, unless the integrand is a square in which case the integral is always equal to half the range of the integral.

Online Problems (Topic 4, questions 1-12)

1. Given a square wave function $f(x) = \begin{cases} f(x) = 1 \text{ for } x = -L/4 \text{ to } L/4 \\ f(x) = 0 \text{ for } x = L/4 \text{ to } 3L/4 \end{cases}$ that repeats every L, show that any integral range is acceptable so long as $L_{max} - L_{min} = L$

2. Find the complex exponential Fourier series for the function specified as $f(x) = e^{-x}$ for $0 < x < 2\pi$

3. Find the Fourier series for the function $f(t) = \begin{cases} 0 \text{ if } -1 \leq t < 1/2 \\ \cos(3\pi \text{ t}) \text{ if } -1 \leq t < 1/2 \\ 0 \text{ if } 1/2 \leq t < 1 \end{cases}$ where the repeat period is 2.

4. Find the Fourier series for the sawtooth function $f(x) = \begin{cases} f(x) = -x \text{ for} - \pi \leq x < 0 \\ f(x) = x \text{ for } 0 \leq x < \pi \end{cases}$

5. Sketch the function $f(t) = \begin{cases} f(t) = 0 \text{ for } t = \text{-4 to } 0 \\ f(t) = 5 \text{ for } t = 0 \text{ to } 4 \end{cases}$ for two periods and find its Fourier series.

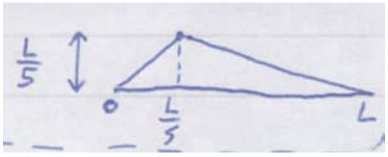
6. The function $f(t)$ with period τ is defined in the interval $[-\tau/2, \tau/2]$ by

$$f(t) = \begin{cases} -1 \text{ if } -\tau/2 < t \le 0 \\ 1 \text{ if } 0 < x < \tau/2 \end{cases}$$

Find the Fourier series.

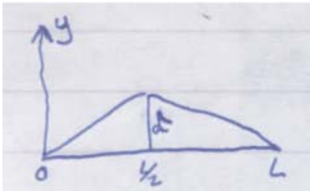
7. I'm thinking of the function $y = |\cos\theta|$. Sketch it, tell me what the period is, and find its Fourier series.

8. A guitarist pulls a string as shown below. Draw the odd and even extensions of the plot and deduce the functions f(x) for both the 0<x<L ranges



9. Find the half range cosine series for question 8 above.

10. Find the Fourier series that represents the displacement y(x) between x=0 and x=L of the string below.



11. Show that the half range cosine series can be used to calculate the Fourier series between 0<x<2π for the sawtooth function in question 4.

12. A function f(x) is defined only on the range 0<x<L and on this range f(x)=k where k is a positive constant. Sketch the function and also show that the half range Fourier sine series of the function is $f(x) = \frac{4k}{\pi}\sum_{n\,odd}\frac{1}{n}\sin\frac{n\pi x}{L}$

Topics 1-4 Summary

Binomial series	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \binom{n}{k}x^n + \ldots$ where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
Taylor series	$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \ldots + \frac{f^n(a)}{n!}x^n + \ldots$ where $f^n(a) = \left.\frac{d^n f}{dx^n}\right\|_{x=a}$
Trig functions	$\cos\alpha x = \frac{1}{2}\left[e^{i\alpha x} + e^{-i\alpha x}\right]$ $\sin\alpha x = \frac{1}{2i}\left[e^{i\alpha x} - e^{-i\alpha x}\right]$ $\cos(-\alpha x) = \cos(\alpha x)$ Even $\sin(-\alpha x) = -\sin(\alpha x)$ Odd $e^{ikx} = \cos kx + i\sin kx$ $e^{-ikx} = \cos kx - i\sin kx$ $(\cos kx + i\sin kx)^n = \cos nkx + i\sin nkx$
Hyperbolic functions	$\cosh\alpha x = \frac{1}{2}\left[e^{\alpha x} + e^{-\alpha x}\right]$ $\sinh\alpha x = \frac{1}{2}\left[e^{\alpha x} - e^{-\alpha x}\right]$ $\cosh\alpha x = \cosh(-\alpha x)$ Even $\sinh\alpha x = -\sinh(-\alpha x)$ Odd $\frac{d(\cosh x)}{dx} = \sinh x$ $\frac{d(\sinh x)}{dx} = \cosh x$
Exponentials & logarithms	$e^{a+b} = e^a e^b$ $e^{-a} = 1/e^b$ $y = e^x \qquad x = \ln y$ $\ln y_1 - \ln y_2 = \ln(y_1/y_2)$ and $\ln y_1 + \ln y_2 = \ln(y_1 y_2)$ $\ln y = \ln\left(10^w\right) = w\ln(10) \qquad w = \frac{\ln y}{\ln(10)}$ or $y = e^{w\ln(10)}$.
Complex numbers	$z = a + ib$ $i = \sqrt{-1}$ $z^* = a - ib$ $\|z\| = \sqrt{zz^*} = \sqrt{a^2 + b^2}$ $z = a + ib = re^{i\phi}$ $r = \sqrt{a^2 + b^2}$ $\tan\phi = \frac{b}{a}$ $z^n = r^n e^{in\phi}$ $z^{1/n} = r^{1/n}e^{i(\phi+2p\pi)/n}$ where there are n distinct roots
2nd order ODEs	$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ has trial solution $x = e^{mt}$ leading to auxiliary equation $am^2 + bm + c = 0$ with general solution $x = Ae^{m_1 t} + Be^{m_2 t}$
Inhomogeneous 2nd order ODEs	$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$ has solution $x = x_c + x_p$ where x_c is the solution to the related homogenous equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ and x_p is a particular solution of found using an appropriate trial solution.
Fourier series	Any periodic function (period 2π) can be written $f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$ where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$; $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$; $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$
Complex Fourier series	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$
Orthogonality	$\int_0^{2\pi} \sin nx\cos mx\,dx = 0$ for $n \neq m$ $\int_0^{2\pi} \sin nx\sin mx\,dx = 0$ for $n \neq m$. $\int_0^{2\pi} \cos nx\cos mx\,dx = 0$ for $n \neq m$