PHY226

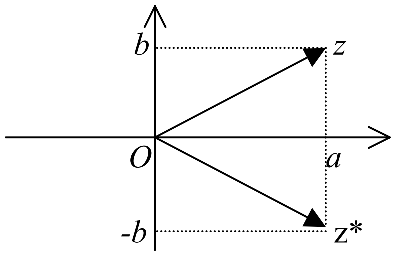
Topic 2 Revision of complex numbers (lecture 2)

Topic 2. Complex Numbers

2.1 Argand diagram

Let $z = a + ib$ where $i^2 = -1$

To represent this number on an **Argand diagram**, plot the point with Cartesian coordinates (a, b). i.e. real numbers run along the x axis and imaginary numbers along the y axis.



The **complex conjugate** is $z^* = a - ib$

(Note: Physicists usually use i, engineers often use j and physicists *always* denote complex conjugates by z* *not* $\bar{z}$)

By Pythagoras, the length OZ is $r = \sqrt{a^2 + b^2} = |z|$.
Note that this length is also equal to $\sqrt{zz^*}$

$zz^* = (a + ib)(a - ib) = a^2 + (ib)(-ib) = a^2 + b^2$.

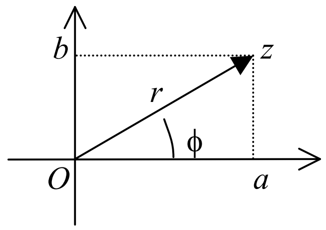
We also write $a^2 + b^2 = zz^* = |z|^2$

where $|z| = \sqrt{zz^*}$ and is called the modulus of z.

Example 2.1 Find the modulus of $|(2 + 3i)|$?

2.2 Polar form

We can also write



$z = re^{i\phi} = r(\cos\phi + i\sin\phi)$ where $0 < \phi < 2\pi$

$r =$ is again called the modulus,
ϕ is called the *argument* or *phase*.

For a proof of this relationship see Lecture 2, problem 2 in the Online Problems.

Then $z^* = re^{-i\phi} = r(\cos\phi - i\sin\phi)$
So $zz^* = r^2 e^{i\phi} e^{-i\phi} = r^2$ since $e^{i\phi}e^{-i\phi} = e^{i(\phi-\phi)} = 1$
Clearly $a = r\cos\phi$, $b = r\sin\phi$ and $r = \sqrt{a^2 + b^2} = |z|$

Note that $|z|$ and $zz^* = |z|^2$, are always *real*, whereas $z^2 = a^2 + 2iab - b^2 = r^2e^{2i\phi} \neq |z|^2$ is usually *complex*. In physics we *always* need to get real answers, hence in quantum mechanics etc. one takes $|\psi|^2$ not ψ^2. (In optics and E&M you may sometimes take the real part to get your answer.)

Changing between the forms $z = a + ib$ and $z = re^{i\phi}$

You are strongly advised to first *plot the number on an Argand diagram*. Without this it is easy to make mistakes about minus signs and angles, etc.!

Given $z = a + ib$, to find the form $z = re^{i\phi}$
Find r using $r = \sqrt{a^2 + b^2}$.
Find ϕ using $\tan\phi = \frac{b}{a}$ (specifying its sign from the quadrant of the Argand diagram.)
Given $z = re^{i\phi}$, to find the form $z = a + ib$ is easier: $a = r\cos\phi$ and $b = r\sin\phi$.

Why do we need both forms?

It is easier to *add* and *subtract* complex numbers in the form $z = a + ib$ but easier to *multiply, divide,* take *powers* and *roots* when they are in the form $z=re^{i\phi}$.
In physics we almost always use the form $z = re^{i\phi}$.

Addition and Subtraction

If $z = a + ib$ and $w = c + id$

then $z + w = (a + c) + i(b + d)$ and $z - w = (a - c) + i(b - d)$.

Multiplication and Division

For this we *always* use the form $z = r\,\mathrm{e}^{\,\mathrm{i}\phi}$

Let $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$
then $z_1 z_2 = r_1 e^{i\phi_1} r_2 e^{i\phi_2} = r_1 r_2 e^{i(\phi_1 + \phi_2)}$.

i.e. *multiply* the moduli and *add* the arguments (phases).

Similarly for division:

$(z_1 / z_2) = (r_1 / r_2)\, e^{i(\phi_1 - \phi_2)}$..
i.e. we *divide* the moduli and *subtract* the arguments.

Example 2.2 Express (1 + i) ÷ (1 + 1.73i) in polar coordinates?

2.3 Powers and Roots

Again we *always* use the polar form. For a real number power it is straightforward:

$$z^n = r^n e^{in\phi}$$

i.e. we take the modulus to the n^{th} power and multiply the argument (or phase) by *n*.

Roots are trickier. We defined ϕ to lie in the region $0 < \phi < 2\pi$. But this will need to be extended if we want to get *all* the roots of a complex number.

We define $z = re^{i(\phi + 2p\pi)}$ where *p* is an integer.

To find an n^{th} root, we need to take *n* distinct values of *p*: $p = 0,\ p = 1, p = 2,\ \ldots,\ p = n - 1$.

Then there are *n* distinct roots to $z^{1/n} = r^{1/n} e^{i(\phi + 2p\pi)/n}$.

<u>**Example 2.3**</u> : If $z = 9\ e^{i\pi/3}$ what is $z^{½}$?

Step 1:

write down z in polars with the $2\pi p$ bit added on to the argument. $z = 9e^{i\ (\pi/3\ +\ 2\pi p)}$

Step 2:

say how many values of p you'll need and write out the rooted expression here $n = 2$ so I'll need 2 values of p; $p = 0$ and $p = 1$ $z^{½} = \sqrt{9}\ e^{i\ (\ \pi/3\ +\ 2\pi p)/2}$

Step 3:

Work it out for each value of p: $z^{½} = 3e^{i\ (\pi/3)/2} = 3e^{i\ (\pi/6)}$ for $p = 0$

$z^{½} = 3e^{i\ (\pi/3\ +\ 2\pi)/2} = 3e^{i\ (\pi/6\ +\ \pi)}$ for $p = 1$

There are your answers but remember that $e^{i\phi} = (cos\phi + isin\phi)$ so $e^{i\pi} = -1$

It's therefore better to write $z^{½} = 3e^{i\ (\pi/6\ +\ \pi)} = 3e^{i\ \pi/6}(e^{i\pi}) = -3e^{i\ \pi/6}$ for $p = 1$, and $3e^{i\ (\pi/6)}$ for $p = 0$

Example 2.4: If $z = 27\,e^{i\pi/2}$ what is $z^{1/3}$?

***Step 1*:** write down z in polars with the $2\pi p$ bit added on to the argument.

***Step 2*:** say how many values of p you'll need and write out the rooted expression

***Step 3*:** Work it out for each value of p

2.4 Exponentials and Trigonometric functions

$$e^{ikx} = \cos kx + i\sin kx \; ; \text{ and } \; e^{-ikx} = \cos kx - i\sin kx$$

Rearranging gives $\cos kx = \frac{1}{2}(e^{ikx} + e^{-ikx})$; $\sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx})$

This is a key observation…remember this.

2.5 Differentiation of a Complex Exponential

We know $\frac{d}{dx}e^{kx} = ke^{kx}$. Since i is just a constant, we similarly have $\frac{d}{dx}e^{ikx} = ike^{ikx}$

Note that is much nicer to differentiate exponentials than sines and cosines because we get exactly the same function as we started with, just multiplied by a constant.

Online problems for (Topic 2, questions 1-11)

1. Expand $\sinh x$ using the Taylor series up to the 5th power term in x

2. Prove the relationship $z = re^{i\phi} = r(\cos\phi + i\sin\phi)$ using the Taylor expansion

3. Find the modulus of i) $6+7i$ ii) $\frac{6+7i}{3+2i}$ and iii) plot $(6+7i)-(3+2i)$ on an Argand diagram and give the answer in polar co-ordinates. iv) Express $z = 5e^{i\pi/6}$ in Cartesian co-ordinates and v) Convert $3+4i$ and $5+12i$ into polar co-ordinates and multiply them.

4. In quantum mechanics we sometimes need to evaluate the modulus squared of the sum of two complex numbers. If $z_1 = Ae^{i\alpha}$ and $z_2 = Be^{-i\beta}$ find $|z_1 + z_2|^2$ Manipulate your answer into the form $A^2 + B^2 + AB\cos(\alpha - \beta)$

5. $z = 2+3i$ By changing into polar co-ordinates find z^{18}?

6. $z = 16e^{i\pi/4}$ Find $z^{1/4}$?

7. Show that $\left|\frac{a+ib}{a-ib}\right|^2 = 1$ for any real numbers a and b

8. Find $\sqrt{i}$

9. $z = 1+i$ Find $z^{1/2}$

10. Prove $x = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$ can be written as $x = e^{\alpha t}(C\sin\beta t + D\cos\beta t)$ and define C and D in terms of A and B

11. Prove that $x = e^{\alpha t}(C\sin\beta t + D\cos\beta t)$ can be written as $x = Ee^{\alpha t}(\cos(\beta t + \phi))$ and define ϕ and E in terms of C and D.