2nd sheet of tutorial problems in special relativity, PHY206

1. A 660 keV γ–ray scatters elastically off a stationary electron. The γ–ray after the scattering has an energy of 550 keV.
(a) What is the scattering angle in the lab frame?
(b) What is q^2 for the electron, in $(\mathrm{keV/c})^2$?
(c) What would the maximum possible q^2 in the collision have been, if the scattering angle can be anything we like?
(d) Find the four components of the 4–momentum transfer q to the recoiling electron in the lab frame of the incoming electron for the case of maximum q^2 from part (c). Give the components in keV/c.
(e) How fast is the centre of mass frame moving with respect to the lab frame for the incident photon approaching the target? Recall that the centre of mass frame is the frame where the momenta of the photon and target electron are equal and opposite.
(f) Let the primed frame be the centre of mass frame. Work out q', again for the case of maximum q^2.
(g) Verify that the square of the 4 momentum transfer is the same in the lab and the centre of mass frames, so that $q^2 = q'^2$.

Solutions to tutorial problems set 2, PHY206 relativity

1a. The formula connecting scattering angle and the initial and final state energies of the gamma is the Compton scattering formula, which was Equation 20 in Lecture 3. The formula is

$$\frac{(E_\gamma^i - E_\gamma^f)}{E_\gamma^i E_\gamma^f} = \frac{1-\cos\theta}{m_0 c^2}.$$

Here E_γ^i and E_γ^f are the gamma energies before and after the collision, and m_0 is the mass of the electron, which is $511\,\mathrm{keV/c^2}$. Using this formula is easiest if we express all energies in keV. The rest energy of the electron is $m_0c^2 = 511\,\mathrm{keV}$. Substituting into this formula leads to
$1-\cos\theta = (660\mathrm{keV} - 550\mathrm{keV}) \times 511\mathrm{keV}/(660\mathrm{keV} \times 550\mathrm{keV}) = 0.154$. Hence $\theta = 32.3°$.

1b. In lecture 6 we learned that in an elastic collision off a stationary target, the 4–momentum transfer squared, q^2 is equal to $2M\nu$, where M is the mass of the stationary target particle and ν is the energy transfer. Here the energy transfer to the target is 660-550=110keV. So $q^2 = 2 \times 511\mathrm{keV/c^2} \times 110\mathrm{keV} = 112,000\,(\mathrm{keV/c})^2$.

1c. Maximal energy transfer corresponds to the case where $\theta = 180°$, so that the γ–ray rebounds back down its incident path. In this case the final state energy is again given by the Compton formula, where $\cos\theta = -1$, so that $1-\cos\theta = 2$. Substituting this in to the

Compton formula and rearranging, we obtain

$$E_{\gamma}^{f} = \frac{E_{\gamma}^{i}}{1 + 2E_{\gamma}^{i}/(m_0c^2)}.$$

Substituting into this we obtain $E_{\gamma}^{f} = 660\text{keV}/(1 + 2 \times 660\text{keV}/511\text{keV}) = 184\text{keV}$. Therefore the energy transfer to the electron is $\nu = 660\text{keV} - 184\text{keV} = 476\text{keV}$. Therefore, again using $(qc)^2 = 2M\nu$, we obtain $q^2 = 2 \times 511\text{keV} \times 476\text{keV/c}^2 = 486,000(\text{keV/c})^2$.

1d. The energy transferred to the electron is already calculated - it's 476keV. The momentum imparted to the electron is the difference between the initial and final state momenta of the γ–ray, which is 660keV/c - (-184keV/c) = 844 keV/c. Note the critical '-' sign. Therefore, for the target electron, $q = (476\text{keV/c}, 844\text{keV/c}, 0\text{keV/c}, 0\text{keV/c})$. In the case of a head on collision, only the x–component of the momentum transfer is non–zero.

1e. Let the primed frame be the centre of mass frame. The Lorentz transformation for momentum is $p' = \gamma(p - \beta E/c)$. For the incoming γ–ray, this becomes ${p_{\gamma}^{i}}' = \gamma(E_{\gamma}^{i}/c - \beta E_{\gamma}^{i}/c)$. For the target electron, this becomes $p'_e = -\beta\gamma m_0 c$, since in the unprimed frame the electron has initially stationary. Setting these momenta to be equal and opposite, we obtain $\beta\gamma m_0 c = \gamma(E_{\gamma}^{i}/c - \beta E_{\gamma}^{i}/c)$. We cancel the γ and arrive at $\beta m_0 c^2 = E_{\gamma}^{i} - \beta E_{\gamma}^{i}$. Solving for β we arrive at $\beta = E/(E + m_0c^2)$, or in the case here $\beta = 660\text{keV}/(660\text{keV} + 511\text{keV}) = 0.563.$. So the centre of mass frame is moving at 0.563c to the right with respect to the lab frame.

1f. For $\beta = 0.563$, we are in the mildly relativistic regime, hence $\gamma = 1/\sqrt{(1 - \beta^2)} = 1.21$. Therefore we transform q as follows: $q'^0 = \gamma q^0 - \beta\gamma q^1 = 1.21 \times 476\text{keV} - 1.21 \times 0.563 \times 844\text{keV}$, or $q'^0 = 1.0\text{keV}$. For the 1st component, $q'^1 = \gamma q^1 - \beta\gamma q^0 = 1.2(844\text{keV} - 0.563 \times 476\text{keV}) = 697\text{keV}$. Therefore we have $q' = (1\text{keV/c}, 697\text{keV/c}, 0\text{keV/c}, 0\text{keV/c})$.

1g. The square of q is $-476^2 + 844^2 = 486,000\,(\text{keV/c})^2$.
The square of q' is $-1^2 + 697^2 = 486,000\,(\text{keV/c})^2$. This verifies that $q^2 = q'^2$.