# Lecture 7 - Rapidity and Pseudorapidity

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## 1 Review of Lecture 6

Last time we studied the 4–momentum transfer when two bodies scatter off each other. If the incident particles have 4–momenta $\mathbf{p_1}$ and $\mathbf{p_2}$, and these scatter off each other in some way, yielding bodies with 4–momenta $\mathbf{p_3}$ and $\mathbf{p_4}$, then where body 1 is identified with body 3, the 4–momentum transfer is $\mathbf{q} = \mathbf{p_3} - \mathbf{p_1}$, which is the same thing as $\mathbf{p_2} - \mathbf{p_4}$. The quantity $q^2$ is Lorentz invariant, that is, it takes the same value when computed by any non-accelerating observer.

We next studied a special case, elastic scattering of a body off an initially stationary target of mass $M_t$ in the lab. We defined the energy transfer $\nu$ as the difference between the final state energy of the target after it is struck and its initial rest energy, $M_tc^2$, or equivalently the energy lost by the incident particle due to the collision. We discovered that $q^2$ is related to $M_t$ and $\nu$ for an elastic collision by

$$q^2 = 2M_t\nu \tag{1}$$

In the last lecture I mistakenly wrote $(qc)^2 = 2M_t\nu$, but this is wrong because the units on the right are not energy squared, but energy times mass. If we did want an expression for $qc$, this would be $(qc)^2 = 2M_tc^2\nu$. I apologise for this mistake. Furthermore Vitaly has been reading my notes, and he has pointed out that the features on the plot of cross section versus $W$ that I showed from the deep inelastic scattering experiments last time are not in fact elastic scattering off constituent quarks. So, I am going to talk a bit more about this stuff first, correct my mistake from last time, and then move on to discuss a new topic. I'm sorry to introduce mistakes in a lecture; my only excuse is that this is the first time I've taught this material. A set of corrected lecture notes from last time has been emailed out to all of you.

After deriving $q^2 = 2M_t\nu$ for elastic scattering, we asked the question, what would we see if the scattering was off a constituent of the target particle, when the target particle is something composite. This depends strongly on how the particle is constituted. But, suppose the particle actually consists of relatively low momentum constituents without too strong of a binding energy to each other. In this case, we might expect to see the scattering equation become $q^2 = 2xM_t\nu$, where $xM_t$ is the mass of the constituent off which the incident particle is scattering, pseudo–elastically. The diagram in the previous set of notes illustrates this. However, how do we know this is happening in an experiment? Suppose we build a detector that measures the by-products of multiple collisions, each time measuring $q^2$ and $\nu$, and calculating the parameter $x$. Next, a histogram is constructed where the number of events as a function of $x$ is plotted. What you might expect to see is a peak at $x = 1$ due to the scattering off the incident projectile off the whole target, and perhaps other peaks for the scattering of the incident projectile off well defined constituents.

A good example of a process where this is observed is in the scattering of electrons off nuclei, for example ${}^4_2\text{He}$, where the incident electron energy is a few hundred MeV. Figure 1 shows the cross section as a function of the energy $E'$ of the scattered electron.

The same effect can be seen when you scatter higher energy electrons off individual protons. Figure 2 shows the cross section as a function of the energy $E'$ of the scattered electrons for electrons scattering off a stationary target containing protons at DESY. Once again, there is a high energy elastic scattering peak. The peaks at lower energies are due to scattering off resonances - particles formed temporarily within the nucleus. These three resonances are the $\Delta(1232)$, $N(1450)$, and $\Delta(1688)$.

It turns out that, though one can infer the existence of quarks by studying the inelastic scattering of electrons off a proton target (hydrogen gas), the quarks have such high Fermi velocity that these peaks are smeared out, and the analysis to infer their existence is more subtle than I thought. If you want to read about it, Perkins's book, 'An Introduction to High Energy Physics', 3rd edition, has a nice discussion in sections 8.2 and 8.3. Figures 1 and 2 are taken from this secondary source. The plot that I showed in the last lecture of results from Friedman, Kendall and Taylor, has resonances in it rather like those in Figure 2. These peaks are due to resonant particle production, not elastic scattering off constituent quarks as I mistakenly claimed last time.

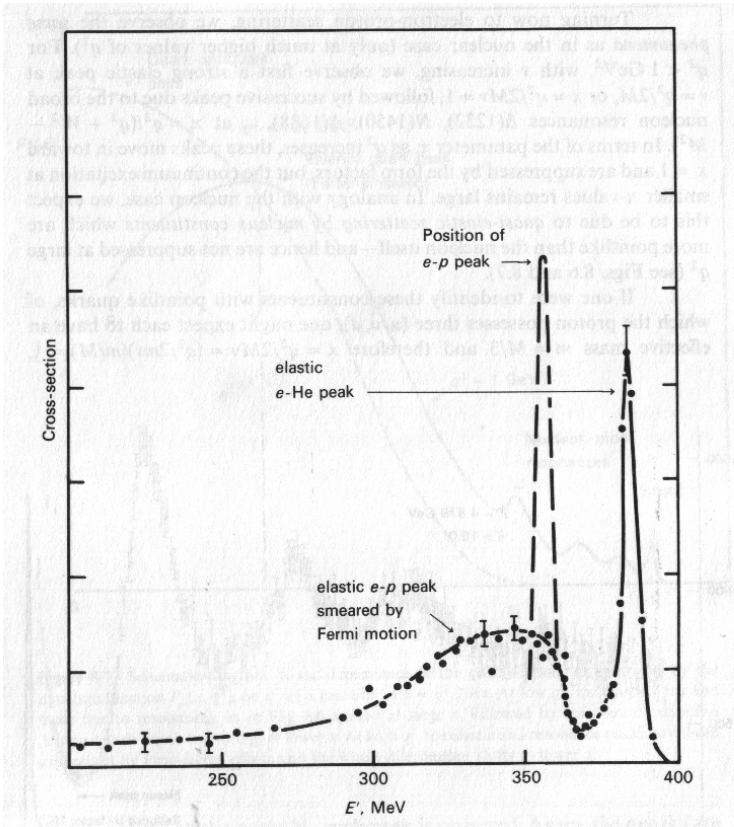


**Figure 8.5** (b) Scattering of 400-MeV electrons in helium at 45°, showing a sharp elastic helium peak and a smeared peak due to elastic scattering from individual nucleons. (After Hofstadter 1956.)

Figure 1: Cross section as a function of scattered energy for 400 MeV electrons incident on a stationary helium target. Note the peak at high energy, corresponding to $x = 1$. There is a less well defined bump between 300 and 360 MeV due to scattering of the electrons off individual protons in the nucleus. The smearing is due to the Fermi momentum of the nucleons; they are far from being stationary, and this smears the final state energy of the electrons after they have struck.

## 2 Rapidity

As our next topic, I wish to start discussing two variables that are in common use in accelerator physics, which derive from the fact that in accelerators the incident velocities of the particles taking part in a collision are along the beam axis. This leads to the definition of various quantities that are either with respect to boosts to the rest frames of observers moving at different velocities parallel to the beam axis, or others that although they are not invariant have transformation properties that are easy to handle and useful for analysis.

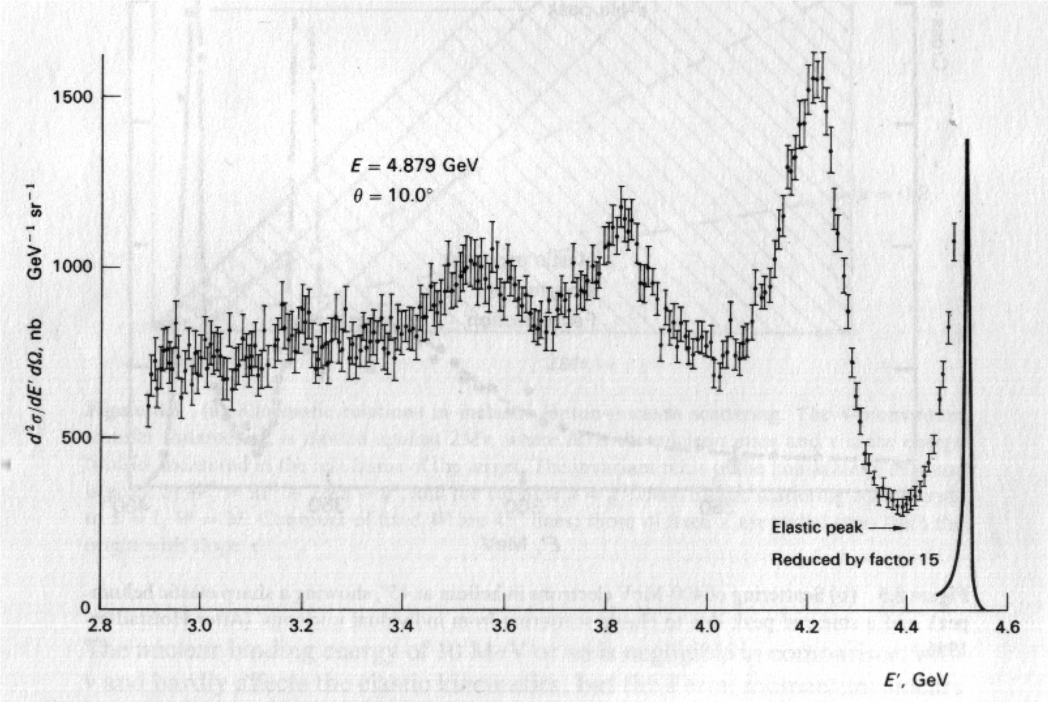


Figure 2: Cross section as a function of scattered energy for 4.88 GeV electrons incident upon a proton target. Note the resonances at lower $E'$ than that of the elastic electron–proton scattering peak.

Firstly, start with the energy–momentum–mass relation for a particle of rest mass $M$.

$$E^2 = p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + M^2c^4 \tag{2}$$

The usual convention in accelerator physics is to take the beam axis as the z–axis, so we will be considering quantities as they appear to observers who are Lorentz boosted with respect to the $z$ axis, not the $x$ axis as we have so far been assuming. There is no difficulty here as long as you write the correct Lorentz transformations. For the components of a 4–displacement,

$$\begin{aligned} ct' &= \gamma(ct - \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - \beta ct). \end{aligned} \tag{3}$$

For the components of a 4–momentum

$$\begin{array}{rcl} E'/c & = & \gamma(E/c - \beta p_z) \\ p'_x & = & p_x \\ p'_y & = & p_y \\ p'_z & = & \gamma(p_z - \beta E/c). \end{array} \qquad (4)$$

Since the $x$ and $y$ components of momentum of a particle, and also its rest mass, are all invariant with respect to boosts parallel to the $z$ axis, we gather up these quantities and define them collectively as the transverse mass $M_T$ (not to be confused with the target mass $M_t$ in the last section and lecture).

$$M_T^2 c^4 = p_x^2 c^2 + p_y^2 c^2 + M^2 c^4. \qquad (5)$$

You may consider it a little strange that we may want to give symbols to quantities who are only invariant with respect to a set of observers whose velocities are all parallel to a single $z$ axis. What is special about these observers? In an accelerator, one is often colliding together particles whose momentum is not equal and opposite, but whose directions are down a common beam $z$ axis. In this case, the centre of mass frame is moving at some velocity down the $z$ axis, so you will often wish to study physics in this frame. However, if you are stuck in the lab frame, you are boosted with some velocity $v_z = \beta c$ with respect to this frame, and the direction of the boost is parallel to the beam axis.

As well as the transverse mass, we also define a quantity called the rapidity, $y$. The definition of the rapidity of a particle is:

$$y = \frac{1}{2} \ln \left( \frac{E + p_z c}{E - p_z c} \right). \qquad (6)$$

Why would you want to define such a quantity? Well, suppose we are dealing with a very high energy product of a collision, in the highly relativistic regime. Suppose now this particle is directed essentially in the $XY$ plane, perpendicular to the beam direction. Then $p_z$ will be small, and the rapidity will be close to 0, because you get the $\log 1$. Now let the same highly relativistic particle be directed predominantly down the beam axis, say in the $+z$ direction. In this case, $E \simeq p_z c$, and $y \to +\infty$. Similarly, if the particle is travelling down the -ve beam axis, the $E = -p_z c$, and in this case you get the natural log of a very small number, and $y \to -\infty$. So, the rapidity is zero when a particle is close to transverse to the beam axis, but tends to $\pm\infty$ when a particle is moving close to the beam axis in either direction. It's related to the angle between the $XY$ plane and the direction of emission of a product of the collision.

There are various neat ways of writing the rapidity, making use of what we know about logs. Taking the 1/2 into the ln and making it a power, you can write $y$ as

$$y = \ln \sqrt{\frac{E + p_z c}{E - p_z c}} = \ln \left( \frac{E + p_z c}{\sqrt{E - p_z c}\sqrt{E + p_z c}} \right). \qquad (7)$$

Using the energy–momentum–mass relation this becomes

$$y = \ln\left(\frac{E + p_z c}{\sqrt{E^2 - p_z^2 c^2}}\right) = \ln\left(\frac{E + p_z c}{M_T c^2}\right). \tag{8}$$

The next neat expression for rapidity is found by using hyperbolic tangents. Recall that $\tanh\theta = (e^\theta - e^{-\theta})/(e^\theta + e^{-\theta})$. We write

$$\begin{aligned}
y &= \tanh^{-1}\left(\tanh\left(\ln\left(\tfrac{E+p_z c}{M_T c^2}\right)\right)\right). \\
&= \tanh^{-1}\left(\frac{\exp\left(\ln\frac{E+p_z c}{M_T c^2}\right) - \exp\left(-\ln\frac{E+p_z c}{M_T c^2}\right)}{\exp\left(\ln\frac{E+p_z c}{M_T c^2}\right) + \exp\left(-\ln\frac{E+p_z c}{M_T c^2}\right)}\right) \\
&= \tanh^{-1}\left(\frac{\frac{E+p_z c}{M_T c^2} - \frac{M_T c^2}{E+p_z c}}{\frac{E+p_z c}{M_T c^2} + \frac{M_T c^2}{E+p_z c}}\right) \\
&= \tanh^{-1}\left(\frac{\frac{(E+p_z c)^2 - M_T^2 c^4}{M_T c^2 (E+p_z c)}}{\frac{(E+p_z c)^2 + M_T^2 c^4}{M_T c^2 (E+p_z c)}}\right) \\
&= \tanh^{-1}\left(\frac{(E+p_z c)^2 - M_T^2 c^4}{(E+p_z c)^2 + M_T^2 c^4}\right) \\
&= \tanh^{-1}\left(\frac{E^2 + 2Ep_z c + p_z^2 c^2 - M_T^2 c^4}{E^2 + 2Ep_z c + p_z^2 c^2 + M_T^2 c^4}\right) \\
&= \tanh^{-1}\left(\frac{2Ep_z c + 2p_z^2 c^2}{2E^2 + 2Ep_z c}\right) \\
&= \tanh^{-1}\left(\frac{p_z c(E+p_z c)}{E(E+p_z c)}\right) \\
y &= \tanh^{-1}\left(\frac{p_z c}{E}\right).
\end{aligned} \tag{9}$$

Now let us show how rapidity transforms under Lorentz boosts parallel to the $z$ axis. Start with Equation 6 and perform a Lorentz boost on $E/c$ and $p_z$

$$\begin{aligned}
y' &= \tfrac{1}{2}\ln\left(\frac{\gamma E/c - \beta\gamma p_z + \gamma p_z - \beta\gamma E/c}{\gamma E/c - \beta\gamma p_z - \gamma p_z + \beta\gamma E/c}\right) \\
&= \tfrac{1}{2}\ln\left(\frac{\gamma(E/c + p_z) - \beta\gamma(E/c + p_z)}{\gamma(E/c - p_z) + \beta\gamma(E/c - p_z}\right) \\
&= \tfrac{1}{2}\ln\left(\frac{E/c + p_z}{E/c - p_z}\frac{\gamma - \beta\gamma}{\gamma + \beta\gamma}\right) \\
&= \tfrac{1}{2}\ln\left(\frac{E + p_z c}{E - p_z c}\right) + \ln\sqrt{\frac{1-\beta}{1+\beta}} \\
.y' &= y + \ln\sqrt{\frac{1-\beta}{1+\beta}} \\
.
\end{aligned} \tag{10}$$

This can be simplified further by noting that

$$\begin{aligned}
\ln\sqrt{\frac{1-\beta}{1+\beta}} &= \tanh^{-1}\left(\tanh\ \ln\sqrt{\frac{1-\beta}{1+\beta}}\right) \\
&= \tanh^{-1}\left(\frac{\sqrt{\frac{1-\beta}{1+\beta}} - \sqrt{\frac{1+\beta}{1-\beta}}}{\sqrt{\frac{1-\beta}{1+\beta}} + \sqrt{\frac{1+\beta}{1-\beta}}}\right) \\
&= \tanh^{-1}\left(\frac{(1-\beta) - (1+\beta)}{(1-\beta) + (1+\beta)}\right) \\
&= -\tanh^{-1}\beta.
\end{aligned} \tag{11}$$

This means that upon Lorentz transforming parallel to the beam axis with velocity $v = \beta c$, the equation for the transformation on rapidity is a particularly simple one,

$$y' = y - \tanh^{-1} \beta. \tag{12}$$

This particularly simple transformation law for $y$ has an important consequence. Suppose we have two particles ejected after a collision, and they have rapidities $y_1$ and $y_2$ when measured by some observer. Now, let some other observer measure these same rapidities, and obtain $y_1'$ and $y_2'$. the difference between the rapidities in the unprimed frame is $y_1 - y_2$, and in the primed frame it becomes

$$y_1' - y_2' = (y_1 - \tanh^{-1} \beta - (y_2 - \tanh^{-1} \beta)) = y_1 - y_2. \tag{13}$$

Therefore the difference between the rapidities of two particles is invariant with respect to Lorentz boosts along the $z$–axis. This is the key reason why rapidities are so crucial in accelerator physics. Rapidity differences are invariant with respect to Lorentz boosts along the beam axis. Rapidity is often paired with the azimuthal angle $\phi$ at which a particle is emitted, so that the angle of emission of a particle from an interaction point is often given as the coordinate pair $(y, \phi)$. This way, the angular separation of two events, $(y_2 - y_1, \phi_2 - \phi_1)$ is invariant with respect to boosts along the beam axis. Histograms binned in either the angular separation of events or the rapidity separation of events can be contributed to by events whose centre of mass frames are boosted by arbitrary velocities with respect to the rest frame of the detector, the lab frame. The resulting histograms are undistorted by these centre of mass frame boosts parallel to the beam axis, as the dependent variable is invariant with respect to this sub–class of Lorentz boosts.

# 3 Pseudorapidity

The only problem with rapidity is that it can be hard to measure for highly relativistic particles. You need both the energy and the total momentum, and in reality it is often difficult to get the total momentum vector of a particle, especially at high values of the rapidity where the $z$ component of the momentum is large, and the beam pipe can be in the way of measuring it precisely. However, there is a way of defining a quantity that is almost the same thing as the rapidity which is much easier to measure than $y$ for highly energetic particles. This leads to the concept of pseudo–rapidity $\eta$.

We start from the definition of $y$,

$$
\begin{aligned}
y &= \tfrac{1}{2}\ln\left(\frac{E+p_zc}{E-p_zc}\right) \\
&= \tfrac{1}{2}\ln\left(\frac{(p^2c^2+m^2c^4)^{\frac{1}{2}}+p_zc}{(p^2c^2+m^2c^4)^{\frac{1}{2}}-p_zc}\right).
\end{aligned} \tag{14}
$$

Knowing that for a highly relativistic particle, $pc$ is far bigger than $mc^2$, we factor $pc$ out of each square root and use a binomial expansion to approximate what is left inside.

$$
\begin{aligned}
y &= \tfrac{1}{2}\ln\left(\frac{pc\left(1+\frac{m^2c^4}{p^2c^2}\right)^{\frac{1}{2}}+p_zc}{pc\left(1+\frac{m^2c^4}{p^2c^2}\right)^{\frac{1}{2}}-p_zc}\right) \\
&\simeq \tfrac{1}{2}\ln\left(\frac{pc+p_zc+\frac{m^2c^4}{2pc}+\cdots}{pc-p_zc+\frac{m^2c^4}{2pc}+\cdots}\right) \\
&\simeq \tfrac{1}{2}\ln\left(\frac{1+\frac{p_z}{p}+\frac{m^2c^4}{2p^2c^2}+\cdots}{1-\frac{p_z}{p}+\frac{m^2c^4}{2p^2c^2}+\cdots}\right).
\end{aligned} \tag{15}
$$

Now $p_z/p = \cos\theta$, where $\theta$ is the angle made by the particle trajectory with the beam pipe, and hence we have

$$
1+\frac{p_z}{p} = 1+\cos\theta = 1+\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right) = 2\cos^2\frac{\theta}{2}. \tag{16}
$$

Similarly

$$
1-\frac{p_z}{p} = 1-\cos\theta = 1-\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right) = 2\sin^2\frac{\theta}{2}. \tag{17}
$$

Substituting these back into Equation 15 we obtain

$$
y \simeq \frac{1}{2}\ln\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}, \tag{18}
$$

or

$$
y \simeq -\ln\tan\frac{\theta}{2}. \tag{19}
$$

We define the pseudorapidity $\eta$ as

$$
\eta = -\ln\tan\frac{\theta}{2}, \tag{20}
$$

so that for highly relativistic particles, $y \simeq \eta$. Pseudorapidity is particularly useful in hadron colliders such as the LHC, where the composite nature of the colliding protons means that interactions rarely have their centre of mass frame coincident with the detector rest frame, and where the complexity of the physics means that $\eta$ is far quicker and easier to estimate than $y$. Furthermore, the high energy nature of the collisions mean that the two quantities may in fact be almost identical.