Lecture 6 - 4–momentum transfer and the kinematics of two body scattering

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1 Review of Lecture 5

Last time we figured out the physical meaning of the square of the total 4–momentum in two particles about to collide. Where the 4–momentum is written $\mathbf{p_T} = (\mathbf{p_1} + \mathbf{p_2})$, the dot product of $\mathbf{p_T}$ with itself, $\mathbf{p_T} \cdot \mathbf{p_T}$ is equal to $-M_T^2 c^2$. Here M_T is the mass of the heaviest particle that could be produced in the collision. In the centre of mass (CM) frame, where total momentum is zero, you produce the heaviest particle you possibly can if it is at rest, therefore $-E^2/c^2$ is just $-M_T^2 c^2$. Were anything moving, the kinetic energy would be wasted, and could have instead been used to create a heavier particle. Were there more than one particle in the final state, the rest energy available would have to be split between them, again meaning that any single particle has less rest mass than it could have. So a single particle at rest in the centre of mass frame really does define the heaviest thing you can make. In other frames, it is not possible for a single particle to be at rest on its own, since in these frames the total momentum is not zero. So, the calculation of the heaviest possible particle always uses the centre of mass frame. It's the simplest thing to do.

We then went on to do an extreme example of a problem that uses this construction, the calculation of the energy an ultra high energy proton, a primary cosmic ray, would have to possess in order to scatter off a cosmic background photon at a temperature of 3 kelvin to produce a π^+ meson and a neutron. The answer, 3×10^{20} eV, is the GZK cutoff, above which no primary cosmic rays have yet been observed. The search goes on, though, as discovery of so–called trans GZK protons would indicate new physics.

2 Phase space and resonant production

In this example, trans GZK protons produce not one but two particles, the neutron and the π^+, and at the threshold for this process these two particles are at rest in the CM frame. In reality, this is a threshold configuration; real collisions would essentially always require more energy from the proton, and produce final state particles that are moving. This is because of an argument having to do with phase space which you may well be about to come across in solid state physics, when studying the Debye model of specific heat and the density of states!

Think of the momenta of the two final state particles as vectors starting at the origin in a 3 dimensional space of possible momenta. If the momenta are big, there is a big space of possibilities for the direction in which these vectors point. From quantum mechanics, we are lead to think of momentum states, which can be thought of as occupying a finite volume $\hbar^3$ in this momentum space. If the momenta of the final state particles are tiny or zero, the argument goes, the number of possibilities for the position of the end point of the momentum vector of each particle goes to zero, and the process never occurs. This is the first smell we have had of real particle physics, the intersection of relativity and quantum mechanics. The process is kinematically allowed (ie, all conserved quantities are indeed conserved) to make our neutron and π^+ at rest, but quantum mechanics determines the probability of it actually happening, and in this case that probability is zero. But, even with a very small final momentum for the neutron and π^+, this quantum mechanical probability quickly rises.

In other collisions, a single heavy intermediate state is produced, which then decays into two lighter particles, giving up some of its mass energy to the kinetic energy of the decay products. This is what happened in the case of Z production in LEP or at SLAC. An electron and a proton at high energy collide, make a Z boson which is essentially at rest, and this Z boson almost immediately decays, often into two leptons, say $\mu^+ + \mu^-$. Now the mass of a muon is $107\,\mathrm{MeV/c^2}$, so two muons way far less than a Z boson at $90\,\mathrm{GeV/c^2}$. The extra available energy from the Z decay goes in to the momenta of the muons, which are therefore very large. So when a particle decays, the resulting kinetic energy means that the decay products have large momenta, and therefore the phase space of final states for these particles is large.

An important point here, though, is that this type of process is only likely if the energies of the incident electron and positron are tuned so that M_T is equal to the mass of the Z. At other energies, the probability for the process $e^+ + e^- \rightarrow \mu^+ + \mu^-$ is very much smaller. So, if you were to do an experiment where electron and positron beams were collided at a range of centre of mass energies, you would see no events, or just noise, at all M_T except those close to the mass of a particle capable of being produced by an $e^+ - e^-$ collision. At these values of M_T you would see a greatly enhanced production rate of $\mu^+ \mu^-$ pairs.

So, to hunt for new particles, tune the energies of the incident beams, and look for bumps, or resonances, in the production rate. This is called resonant production. It is one of the most important discovery mechanisms for new particles in high energy physics. The most famous discovery by this route was probably the J/ψ meson. This is a bound state of a charm c and an anticharm $\bar{c}$ quark, discovered by Burt Richter and Sam Ting, and their collaborators, in 1974 [1].

3 Kinematics of two body scattering

We are now going to analyse the kinematics of the scattering of two bodies off each other. We are not for now going to worry about the quantum mechanics that determines how likely the process is to occur, we are just going to apply conservation laws, learning what we can about the process from the final state, taking the occurrence of that final state as a given. Figure 1 shows a general two body scattering process. Bodies 1 and 2 collide, producing bodies 3 and 4 in the final state.

First let us make it clear why the total 4–momentum is a conserved quantity. Writing the 4–vectors out explicitly in terms of their components, the total four momentum initially is

$$\mathbf{p_1} = \begin{pmatrix} E_1/c \\ \vec{p_1} \end{pmatrix} + \begin{pmatrix} E_2/c \\ \vec{p_2} \end{pmatrix}. \tag{1}$$

Similarly, we could write out the expression for $\mathbf{p_f}$, the total final state 4–momentum. Let us assume that the total 4–momentum is conserved, so that

$$\begin{pmatrix} E_1/c \\ \vec{p_1} \end{pmatrix} + \begin{pmatrix} E_2/c \\ \vec{p_2} \end{pmatrix} = \begin{pmatrix} E_3/c \\ \vec{p_3} \end{pmatrix} + \begin{pmatrix} E_4/c \\ \vec{p_4} \end{pmatrix}. \tag{2}$$

We can see that the conservation of 4–momentum is just another way of expressing the fact that energy and momentum are both conserved. And, as for energy and momentum, for the 4–momentum to be conserved, the energy and momenta must be measured in the same frame of reference before and after the interaction

However, we could also square the initial and final 4–momenta in Equation 2, and we would obtain an equation expressing the conservation of the square of

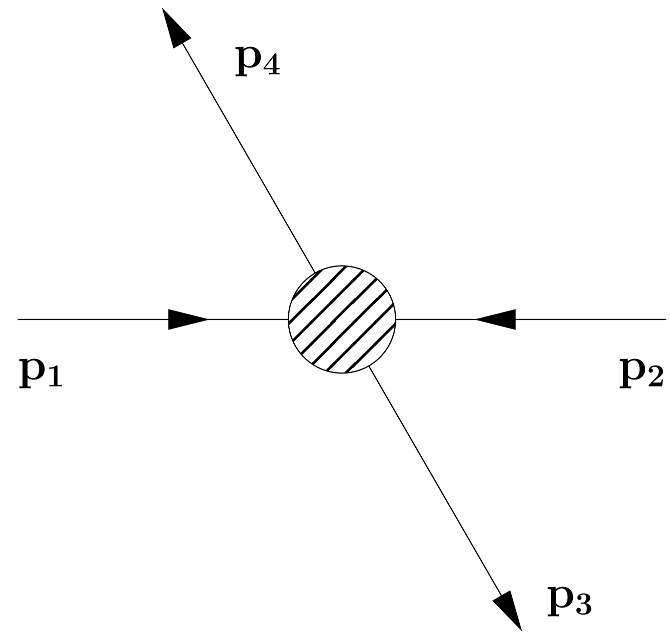

Figure 1: A general two body collision, with the 4–momenta of the incoming particles 1 and 2, and the 4–momenta of outgoing particles 3 and 4, labelled in bold type. No particular reference frame is assumed. This process can represent elastic scattering of two particles off each other, in which case particles 1 and 3 may have the same mass, and similarly for particles 2 and 4. Alternatively, it could represent an inelastic scattering process, with particles 1 and 3 converted into different particles 2 and 4. The shaded circle in the middle signifies that we do not specify at this stage any detailed description of the interaction occuring here.

total 4–momentum. This equation is

$$(\mathbf{p_1} + \mathbf{p_2}) \cdot (\mathbf{p_1} + \mathbf{p_2}) = (\mathbf{p_3} + \mathbf{p_4}) \cdot (\mathbf{p_3} + \mathbf{p_4}). \tag{3}$$

Because the sum of 4–vectors is also a four vector, and the square of any four vector is Lorentz invariant, the dot product of a 4–vector with itself is frame–independent. This combined with the conservation of 4–momentum means that the square of the total 4–momentum is firstly conserved during the interaction, and secondly independent of the velocity of the observer. So one can, as we did last lecture, equate the square of the total 4–momentum before an interaction with all the relevant quantities measured in one reference frame, with the square of the total 4–momentum after the interaction, with all the relevant quantities measured in a different reference frame.

However, today we're going to ask a different question. Are there other 4–vectors apart from the total 4–momentum which we could use to construct interesting invariant quantities? Let's find one of these quantities by rearranging Equation 2 as follows:

$$\mathbf{p_1} - \mathbf{p_3} = \mathbf{p_2} - \mathbf{p_4}. \tag{4}$$

The difference between two 4–vectors is still a 4–vector. Therefore we may take the 4–vector square of each side of this equation and end up with a quantity that is Lorentz invariant.

$$(\mathbf{p_1} - \mathbf{p_3}) \cdot (\mathbf{p_1} - \mathbf{p_3}) = (\mathbf{p_2} - \mathbf{p_4}) \cdot (\mathbf{p_2} - \mathbf{p_4}). \tag{5}$$

Written this way, the equation does not express a conservation law, since on each side of the equation we have a mixture of quantities from before and after the collision process. However, we may still wonder what the physical significance of these quantities are. To find out, choose one side of the equation, and expand the 4–vector dot product.

$$\begin{aligned}(\mathbf{p_1} - \mathbf{p_3}) \cdot (\mathbf{p_1} - \mathbf{p_3}) &= \mathbf{p_1} \cdot \mathbf{p_1} + \mathbf{p_3} \cdot \mathbf{p_3} - \mathbf{2p_1} \cdot \mathbf{p_3} \\ &= -M_1^2c^2 - M_3^2c^2 - 2\begin{pmatrix} E_1/c \\ \vec{p_1} \end{pmatrix} \cdot \begin{pmatrix} E_3/c \\ \vec{p_3} \end{pmatrix},\end{aligned} \tag{6}$$

where in the second line we have explicitly written out the components of each 4–vector and used the equality $\mathbf{p^2} = -E^2/c^2 + |\vec{p}|^2 = -M^2c^2$, by the energy–momentum–mass relation. Now we evaluate the dot product on the right term–by–term and split it into two components for reasons that will become clear presently.

$$(\mathbf{p_1} - \mathbf{p_3}) \cdot (\mathbf{p_1} - \mathbf{p_3}) = -M_1^2c^2 - \frac{E_1E_3}{c^2} - M_3^2c^2 - \frac{E_1E_3}{c^2} - 2\vec{p_1} \cdot \vec{p_3}. \tag{7}$$

Next we write that

$$E_1 - E_3 = \nu, \tag{8}$$

so that in addition $E_1 = E_3 + \nu$ and $E_3 = E_1 - \nu$. In terms of this quantity we have

$$(\mathbf{p_1} - \mathbf{p_3})^\mathbf{2} = -M_1^2c^2 + \frac{E_1(E_1 - \nu)}{c^2} - M_3^2c^2 + \frac{(E_3 + \nu)E_3}{c^2} - 2\vec{p_1} \cdot \vec{p_3}. \tag{9}$$

Next multiply out and rearrange.

$$(\mathbf{p_1} - \mathbf{p_3})^\mathbf{2} = \frac{E_1^2}{c} - M_1^2c^2 + \frac{E_3^2}{c^2} - M_3^2c^2 - 2\vec{p_1} \cdot \vec{p_3} - \frac{E_1\nu}{c^2} + \frac{E_3\nu}{c^2}. \tag{10}$$

Using the energy–momentum–mass relation the pairs of terms in E/c and M to the right of the equals sign may be reexpressed as the squares of $\vec{p_1}$ and $\vec{p_3}$.

$$(\mathbf{p_1} - \mathbf{p_3})^\mathbf{2} = |\vec{p_1}|^2 + |\vec{p_3}|^2 - 2\vec{p_1} \cdot \vec{p_3} - \frac{E_1\nu}{c^2} + \frac{E_3\nu}{c^2}. \tag{11}$$

We may again apply Equation 8 and rearrange to obtain

$$\begin{aligned}
(\mathbf{p_1} - \mathbf{p_3})^{\mathbf{2}} &= -\tfrac{(E_1-E_3)\nu}{c^2} + |\vec{p_1}|^2 + |\vec{p_3}|^2 - 2\vec{p_1}\cdot\vec{p_3} \\
&= -\tfrac{\nu^2}{c^2} + (\vec{p_1} - \vec{p_3})^2
\end{aligned} \qquad (12)$$

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Now we can interpret this equation. The quantity $E_1 - E_3$ is called the energy transfer. It is the amount by which the energy of particle 1 exceeds that of particle 3. To interpret this, consider cases in which 1 and 3 represent the incoming and outgoing tracks of the same particle. In this case, this particle has lost energy ν, and by conservation of energy, it must have been transferred to particle 4 in the final state. This is why it's called the energy transfer. Similarly, $\vec{p_1} - \vec{p_3}$ is the difference in momentum between particle 1 and particle 3. So this could be called (though we will not use this term) the momentum transfer. Overall, the right hand side is the square of a four–vector,

$$\mathbf{q} = \begin{pmatrix} \nu/c \\ p_1 \vec{-} p_3 \end{pmatrix}. \qquad (13)$$

It's components are interpreted as the energy transferred from particle 1 to particle 3, and the momentum shift between particle 1 and particle 3. Its square q^2 is the square of a 4–vector, and hence is Lorentz invariant.

4 Elastic scattering in the lab frame

Figure 2 shows the particular case of elastic scattering that we will consider next, the elastic scattering of a target particle at rest in the lab by some incident particle. In the final state, the target particle, initially at rest, recoils with some energy E_3. The incident energy of the beam particle is E_2.

Let us evaluate the square of the 4–momentum transfer to the stationary target. The energy transfer ν to the stationary target is

$$\nu = E_3 - M_T c^2. \qquad (14)$$

The 4–momentum transfer squared is

$$q^2 = -\frac{\nu^2}{c^2} + |\vec{p_3} - \vec{p_1}|^2. \qquad (15)$$

But $\vec{p_1}$ is zero because the target particle is stationary in the lab. Furthermore, the energy transfer μ is equal to $E_3 - M_T c^2$. Substituting in to Equation 15 we obtain

$$q^2 = \frac{-(E_3 - M_T c^2)^2}{c^2} + |\vec{p_3}|^2. \qquad (16)$$

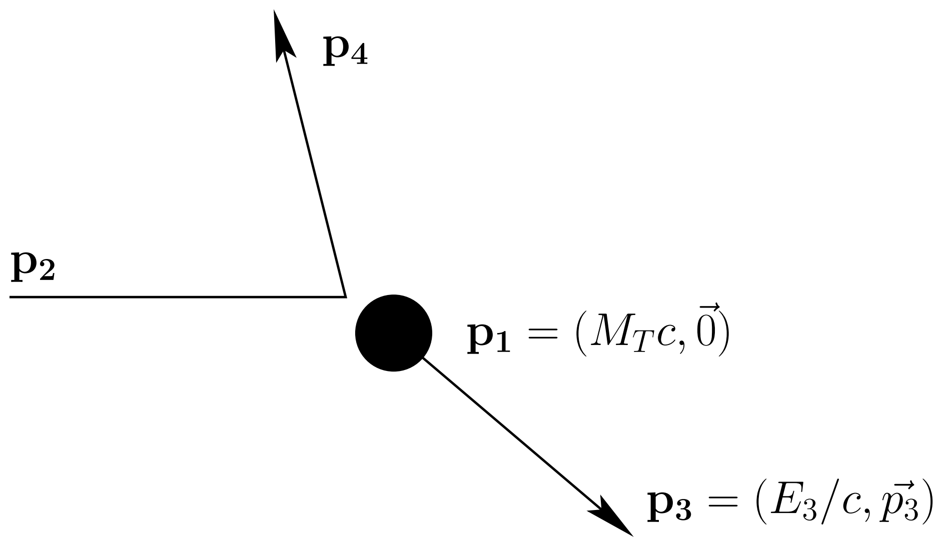

Figure 2: Two body elastic scattering with a fixed target of mass M_T. We have chosen to work in the rest frame of the target.

Next we square out the bracket and collect terms in what will turn out to be a convenient order.

$$q^2 = \frac{-E_3^2}{c^2} + |\vec{p_3}|^2 + \frac{2E_3(M_T c^2)}{c^2} - M_T^2 c^2. \tag{17}$$

Using the energy–momentum–mass relation again, we substitute for the first two terms on the right, arriving at

$$q^2 = \frac{2E_3(M_T c^2)}{c^2} - \frac{2M_T^2 c^4}{c^2}. \tag{18}$$

Finally we note again that $\nu = E_3 - M_T c^2$, so we arrive at

$$q^2 = 2M_T \nu. \tag{19}$$

This equation applies to elastic scattering, because the particle species are unchanged by the collision. It relates an Lorentz invariant quantity, q^2, to another Lorentz invariant quantity, M_T, the rest mass of the target, and the new quantity ν, the energy transfer to the target. Therefore the energy transfer ν must be Lorentz invariant too in elastic scattering processes where this relationship holds.

Next, consider a closely related scattering process, where an incident beam particle, perhaps a point–like particle such as an electron, scatters off a more complex object, perhaps a proton, for example. Some of the time, the target particle will scatter the proton elastically, and Equation 19 applies. However, how do we think about processes in which the incoming electron scatters inelastically in the proton? In this case, perhaps we can consider that the electron

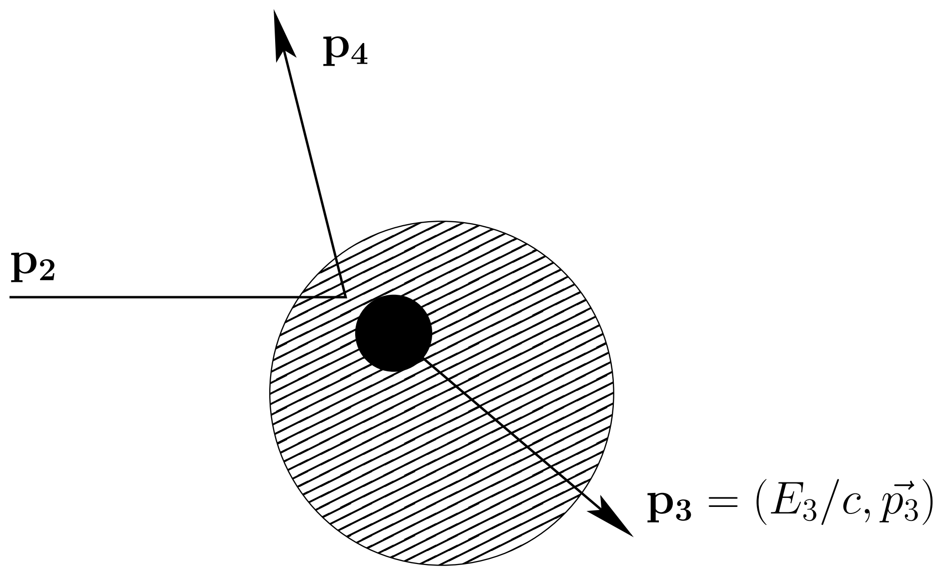

Figure 3: Deep inelastic scattering of an electron in a proton. Although the incident particle does not scatter off the whole nucleus elastically, under some circumstances it can be thought of as scattering effectively elastically off some object within the proton.

is scattering elastically, not off the whole proton, but off some component of the proton. Figure 3 illustrates this type of process.

Now, let us define a variable x, defined as

$$x = \frac{q^2}{2M_T\nu} \tag{20}$$

When this quantity is 1, Equation 19 is satisified, with M_T equal to the mass of the proton, so that the electron is scattering off the whole proton. To check that elastic scattering is all that occurs, we might do the following experiment: construct a fixed target experiment with a proton bearing target and an electron beam incident upon it. For each incident electron, measure its recoil energy and momentum, and hence determine the four momentum transfer to the target proton and its square q^2. Infer also the energy transfer ν, which is the difference between the electron energy before and after the collision. Hence deduce x for the collision, using for M_T the mass of the target proton. As a function of this measured x, plot a histogram of the scattering events.

In the case where protons have elastic scattering only, one might expect the scattering events all to have $x = 1$, and hence the histogram would have a big peak there, and would have far fewer events at x in the intervening range $[0, 1]$.

Next lecture we consider the consequences of elastic scattering off constituents and in particular the production and detection of resonances in the nucleus.

References

[1] R. Cahn and G. Goldhaber, *The experimental foundations of particle physics*, Cambridge University Press, 1989. This is an excellent book, which also contains reprints of all the original papers on the subject. Chapter 9 is on the discovery of the J/ψ. Chapter 8 is on the discovery of quarks within the proton.