

Lecture 5 - Ultra high energy cosmic rays and the GZK cutoff

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1 Review of Lecture 4

Last time we studied use of 4-vectors, particularly the 4-momentum, in relativity calculations. We learned that if a particle has total energy E and momentum \vec{p} , then we may write its 4-momentum as follows:

$$\mathbf{p} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}. \quad (1)$$

The symbol \mathbf{p} in bold means the 4-momentum. The vector arrow above the \vec{p} denotes the ordinary momentum vector, which can be written in terms of its Cartesian components (p^x, p^y, p^z) , or alternatively in terms of components in some other coordinate system. So overall \mathbf{p} is a four component object.

We also learned how to generate an invariant quantity from the 4-momentum. We defined the square of the 4-momentum $\mathbf{p} \cdot \mathbf{p} = \mathbf{p}^2$ as follows:

$$\begin{aligned} \mathbf{p} \cdot \mathbf{p} = \mathbf{p}^2 &= -\left(\frac{E}{c}\right)^2 + \vec{p} \cdot \vec{p}. \\ &= -m_0^2 c^2, \end{aligned} \quad (2)$$

where the second line follows as a consequence of the energy-momentum-mass relation, $E^2 = p^2 c^2 + m_0^2 c^4$. Notice that the formula is not the same as the formula for the dot product of two ordinary three dimensional vectors because of the minus sign preceding the square of the 0th component.

Equation 2 means that the square of the four momenta of individual particles are particularly simple, yielding $\mathbf{p}^2 = -m_0^2 c^2$ for massive particles, and $\mathbf{p}_\gamma^2 = 0$ for photons and other massless particles.

We also learned to write down the total 4-momentum of a system as another 4-component object with the zeroth component containing the total energy (divided by c), and the other three components containing the three components of the total momentum. This means that in the centre-of-mass frame, where the total momentum is zero, the total 4-momentum has only one non-zero component, the zeroth component, the total energy over c ,

$$\mathbf{P}_{\text{CM}}^2 = -\frac{E^2}{c^2}. \quad (3)$$

Now let us think a little more about this business of the total four momentum of a system of particles. If we have several particles, we are free to add up their energies and get the total energy, E_t . We are also free to add up their momenta, so long as we add them as vectors, to obtain the total momentum \vec{p}_t . The t subscript denotes total. We may form a 4-vector out of the total energy and total momentum of the system, just as we did for an individual particle,

$$\mathbf{p}_t = \begin{pmatrix} E_t/c \\ \vec{p}_t \end{pmatrix}. \quad (4)$$

Now, again just as for an individual particle, we may square the four momentum to obtain $\mathbf{p}_t \cdot \mathbf{p}_t = -m_t^2 c^2$, where m_t is some total invariant mass for the system. By invariant, I mean that any inertial observer measuring E_t and p_t , and from these numbers forming \mathbf{p}_t , and then squaring this total 4-momentum will obtain the same result, $-m_t^2 c^2$. What is m_t ? m_t is the maximum mass that this set of particles can produce when they collide, if this particle is produced at rest. $m_t c^2$ is called the centre of mass energy of the system. Sometimes it is given the symbol \sqrt{s} . It's an invariant quantity. Furthermore, because total energy and total momentum are conserved, it's also a conserved quantity when calculated before and after some collision or decay.

Note that the above does not mean that any system of particles is equivalent to a set of particles at rest in the centre of mass frame! Just because a centre of mass frame exists, this does not mean that in that frame all the particles are at rest. It may be that you have, for example, two electrons in the centre of mass frame with equal energy and opposite momenta. Their total momentum is zero, but it is certainly not true that all the energy of this configuration is in rest energy. On the other hand, if these two particles collide, then they might produce a single particle at rest. In fact, this is terribly unlikely because of the so-called quantum mechanical phase space argument. If you are trying to produce a single particle at rest, its momentum must be exactly zero. The number of quantum mechanical states having momentum exactly zero is of order 1. If the number of possible final quantum states is

very small, then the probability of the process occurring is also very small. So the production of a single particle at rest in a centre of mass frame collision is very unlikely. But, that's quantum mechanics; from the point of view of the relativistic dynamics of the collision there is nothing that prevents this process.

The power of the invariant mass, and the squared total 4-momentum that generates it is that you can do dynamics calculations particularly easily, since the square of the total 4-momentum of a system can be evaluated in one reference frame before the collision, and if necessary a more convenient reference frame after the collision. Let's now study an example of a calculation that can make use of conservation of the total squared 4-momentum.

2 Ultra high energy cosmic rays and the GZK cutoff

Now for another example, and this time a very interesting one, the so-called GZK cutoff (GZK stands for Greisen-Zatsepin-Kuzmin). This cutoff has to do with the passage of ultra high energy cosmic rays through intergalactic space. As you know, space is permeated by microwave photons which are the radiation fall-out of the big bang. Often they are called the cosmic microwave background radiation, or CMBR. Sometimes they are just called the cosmic background radiation, which is sensible because they have only become microwaves today due to being redshifted during the expansion of the Universe - when produced they must have had a very much shorter wavelength.

Now, enter an ultra high energy proton, produced in some faraway astrophysical accelerator (next to a black hole accreting matter, for example). It's barrelling through what to most of us would seem like a boring vacuum, on its way towards Earth. However, because it's going so fast, it's total energy is enough to create new particles if it can scatter off something to conserve both energy and momentum. In particular, our high energy proton could potentially scatter off a photon from the CBR. The following process might then occur:



The positively charged pion ensures that charge is conserved. The question is, what energy does the proton have to possess in the lab frame (defined by the rest frame of the cosmic background radiation), for this

process to be possible? This is an important question, because if this process is possible, cosmic rays having energies above this threshold would never reach us, since they would be overwhelmingly likely to scatter off a CBR photon somewhere on their journey and convert into neutrons and pions, thereby never reaching us and being detected.

To start off with, we wish to know the energy of a typical CBR photon. The temperature of the CBR today is approximately 3 kelvin. The energy of a photon having this temperature is

$$\begin{aligned}
 k_B T &= 1.38 \times 10^{-23} [\text{J K}^{-1}] \times 3 [\text{K}] = 4.2 \times 10^{-23} \text{J} \\
 &= 4.2 \times 10^{-23} \text{J} / 1.6 \times 10^{-19} [\text{J eV}^{-1}] \\
 &= 263 \mu\text{eV} \\
 &= 2.63 \times 10^{-10} \text{MeV}.
 \end{aligned} \tag{6}$$

This is a tiny energy compared to that of the proton. Now, in the lab frame, let us write the sum of the 4-momenta of the proton and the gamma, and square it. We then remember that we want the lowest energy the proton can have to react with the CBR photon to produce a neutron and a neutrino. As with the example we did last time, the lowest proton energy that can yield these two particles will produce them both at rest in the centre of mass frame.

Now, I realise that the lab frame is very, very far from the centre of mass frame. That really doesn't matter. We simply equate the square of the total 4-momentum in the lab frame before the collision with the square of the total 4-momentum of the pion and the neutron in their centre of mass frame after the collision. We can do this because the square of the total four momentum is Lorentz invariant; any inertial observer will get the same value for it. So, we obtain

$$(\mathbf{p}_p + \mathbf{p}_\gamma)^2 = (\mathbf{p}_n + \mathbf{p}_\pi)^2. \tag{7}$$

Since at the threshold for production of the pion and neutron, they are both produced at rest, their total energy is $(M_n + M_\pi)c^2$. Since we're in their centre of mass frame, their total momentum is zero. Hence we have, after the collision, at the threshold where there is just enough energy to make a neutron and a pion both at rest in their centre of mass frame,

$$(\mathbf{p}_n + \mathbf{p}_\pi)^2 = -(M_n + M_\pi)^2 c^2 \tag{8}$$

Now to deal with the left hand side of the equation. Let's just take the dot product term-by-term on the left and see what we come up with.

$$\begin{aligned}
 (\mathbf{p}_p + \mathbf{p}_\gamma)^2 &= (\mathbf{p}_p + \mathbf{p}_\gamma) \cdot (\mathbf{p}_p + \mathbf{p}_\gamma) \\
 &= \mathbf{p}_p^2 + 2\mathbf{p}_p \cdot \mathbf{p}_\gamma + \mathbf{p}_\gamma^2,
 \end{aligned} \tag{9}$$

where each of the terms on the right hand side is a 4–vector dot product following the usual rule discussed at the start of this lecture. Next, it is time to invoke some of the rules for squares of four momenta of individual particles. For the proton, we have $\mathbf{p}_p^2 = -M_p^2 c^2$. For the photon, we have $\mathbf{p}_\gamma^2 = 0$. The left hand side therefore simplifies and Equation 8 becomes

$$-M_p^2 c^2 + 2\mathbf{p}_p \cdot \mathbf{p}_\gamma = -(M_n + M_\pi)^2 c^2. \quad (10)$$

Next we evaluate $\mathbf{p}_p \cdot \mathbf{p}_\gamma$. To maximise the energy available from the collision, we make the momenta of the two particles in opposite directions. The 4–momentum of the proton is

$$\mathbf{p}_p = \begin{pmatrix} E_p/c \\ E_p/c \end{pmatrix}. \quad (11)$$

Here we have assumed that we are in the highly relativistic regime, so that for the proton $E_p = p_p c$, so that $E_p/c = p_p$.

Similarly for the photon, we write

$$\mathbf{p}_\gamma = \begin{pmatrix} E_\gamma/c \\ -E_\gamma/c \end{pmatrix}. \quad (12)$$

Therefore the dot product of the proton and photon 4–momenta is

$$\begin{aligned} \mathbf{p}_p \cdot \mathbf{p}_\gamma &= -\frac{E_p E_\gamma}{c^2} + \left(\frac{E_p}{c}\right) \left(\frac{-E_\gamma}{c}\right) \\ &= \frac{-2E_p E_\gamma}{c^2}. \end{aligned} \quad (13)$$

Substituting the dot product back into Equation 10 becomes

$$M_p^2 c^2 + \frac{4E_p E_\gamma}{c^2} = (M_n + M_\pi)^2 c^2. \quad (14)$$

Rearranging and expressing in terms of rest energies by multiplying throughout by c^2 , we obtain

$$4E_p E_\gamma = (M_n c^2 + M_\pi c^2)^2 - (M_p c^2)^2 \quad (15)$$

or

$$E_p = \frac{(M_n c^2 + M_\pi c^2)^2 - (M_p c^2)^2}{4E_\gamma}. \quad (16)$$

Now to insert some numbers. The mass of a neutron is $M_n = 939.6 \text{ MeV}/c^2$, and that of a proton is $M_p = 938.3 \text{ MeV}/c^2$. The mass of a π^+ meson

is $M_\pi = 139.6 \text{ MeV}/c^2$. Using the energy of a CBR photon calculated in Equation 6 we obtain

$$\begin{aligned} E_P &= \frac{(939.6+139.6)^2 - 938.3^2 \text{ [MeV}^2\text{]}}{4 \times 2.6 \times 10^{-10} \text{ [MeV]}} \\ &= 3 \times 10^{20} \text{ eV.} \end{aligned} \quad (17)$$

As a consequence of this limit, we do not expect to detect cosmic rays from deep space having energies greater than roughly 10^{20} eV, because any such cosmic ray particles would scatter off the CBR photons and be lost.

Figure 1 is an energy spectrum of ultra high energy cosmic rays from various instruments. On the horizontal axis is the cosmic ray energy, and on the vertical axis is the measured flux. It is clear that the flux is dropping off to zero at roughly 10^{20} eV as predicted by our scattering calculation.

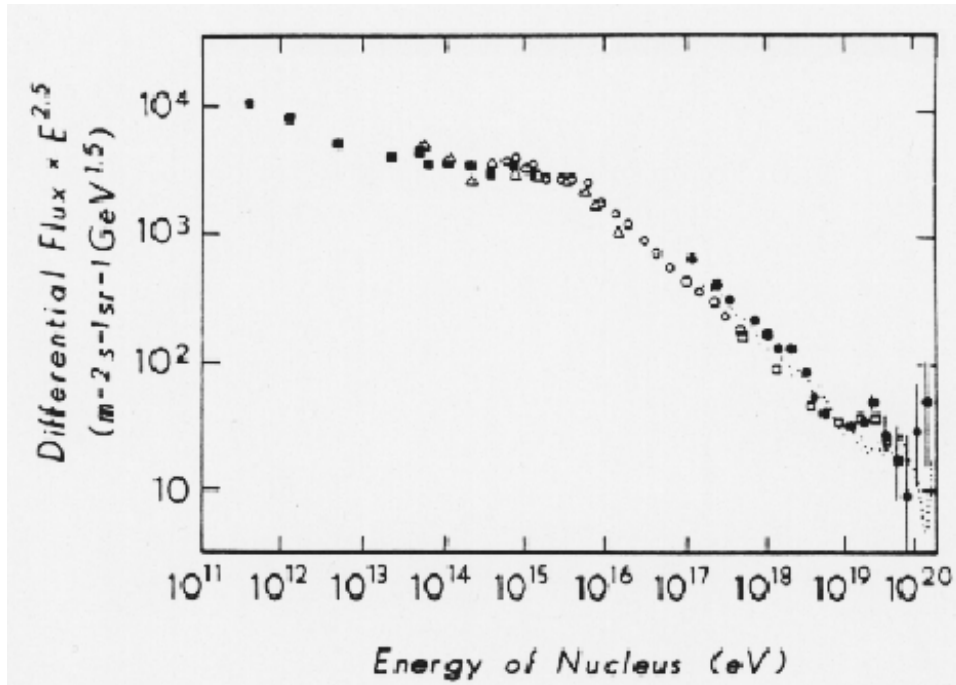


Figure 1: An energy spectrum of ultra high energy cosmic rays, taken from http://imagine.gsfc.nasa.gov/docs/features/topics/snr_group/images/cr-knee.gif The GZK cutoff is visible at the right hand edge of the figure, where the cosmic ray energy spectrum appears to cut off right where absorption of protons by the CBR is predicted to turn on.

This treatment of the GZK cutoff is approximate. In particular, we have not taken into account the statistical spread in cosmic background

photon energies; it turns out that to get the right answer, one has to assume that there is an appreciable number of more energetic photons on the tail of the Maxwell Boltzmann distribution, and it is these photons which will define the GZK cutoff. Also, the process that we have discussed, $p + \gamma \rightarrow \pi^+ + n$ is not the only microwave background scattering process for high energy protons. In particular, a second process, $p + \gamma \rightarrow p + \pi^0$ also takes place, and indeed is energetically preferred because the final state particles are lighter. Though it seems that you get the proton back, energy will have been transferred to the π^0 , and therefore the energy of the proton is greatly reduced, to far below the GZK cutoff. Taking into account these and other details, the energy at which you begin to see suppression of GZK photons is in fact around 3×10^{19} eV.

Next time, we'll do some elastic scattering calculations, and perhaps introduce the useful 4-vector known as the 4-momentum transfer.