Lecture 3 - Compton Scattering

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1 Review of Lecture 2

Last time we recalled that in special relativity, as in pre-relativistic dynamics, the total energy in an interaction or collision is conserved, as is the vector total momentum, as long as the measurements of these quantities are carried out by the same non–accelerating observer before and after the collision. We also recalled from the previous course in year 1 some formulae for the energy and momentum of a moving particle. The total energy E can be written $E = \gamma m_0 c^2$, where m_0 is the rest mass. The magnitude of the momentum can be determined from $pc = \beta\gamma m_0 c^2 = \beta\gamma E_R$, where $E_R = m_0 c^2$ is the energy of a particle at rest. Recall that pc has energy units, so that if $pc = 75\,\mathrm{MeV}$, then the momentum is $p = 75\,\mathrm{MeV/c}$.

2 The Energy–Momentum–Mass relation

The above formulae lead us back to the energy–momentum–mass relation that we saw last year. If we sum the squares of the momentum and pc then we obtain

$$
\begin{aligned}
(pc)^2 + (m_0c^2)^2 &= (\beta^2\gamma^2 + 1)m_0^2c^4 \\
&= \left(\frac{\beta^2}{1-\beta^2} + 1\right) m_0^2c^4 \\
&= \left(\frac{\beta^2+(1-\beta^2)}{1-\beta^2}\right) m_0^2c^4 \\
&= \left(\frac{1}{1-\beta^2}\right) m_0^2c^4 \\
&= \gamma^2 m_0^2c^4 \\
&= E^2,
\end{aligned} \tag{1}
$$

so that

$$E^2 = (pc)^2 + (m_0c^2)^2 \tag{2}$$

This equation is very useful in doing problems where we are using conservation of energy and momentum. A special case of this result is that where the particle has either zero rest mass, or its rest energy is negligible compared to its total energy. In these cases only we may write

$$E = cp \tag{3}$$

This equation is exact for photons and approximate for particles whose total energy is much greater than their rest energy.

Again, these are equations where use of quantities with energy units comes in very handy. Suppose we have a particle whose rest mass is $938\,\mathrm{MeV/c^2}$ (a proton), and it has a total energy of $6\,\mathrm{GeV}$. Then its momentum is given by $pc = \sqrt{6^2[\mathrm{GeV}^2] - 0.94^2[\mathrm{GeV}^2]} = 5.92\mathrm{GeV}$. Therefore its momentum is $5.92\,\mathrm{GeV/c}$. Notice that nowhere did I divide or multiply by 3×10^8. The factors of c are left as just that - symbolic factors of c that make the dimensions work out. The numbers are all electron volts, the factors of c just tell us where we would need to divide or multiply by c to convert back to a momentum or a mass. We should never need to do this conversion if we stick with quantities in electron volts.

3 Compton Scattering

Compton scattering is a very important process in physics. It is the scattering of a photon off a charged particle. Considering the charged particle to be initially at rest to some observer, in the final state the particle moves off with some kinetic energy. Therefore the energy of the scattered photon must have dropped. Consequently, Compton scattering is one of the primary mechanisms for the loss of energy of gamma ray photons when they pass through matter.

Figure 1 is a diagram of Compton scattering.

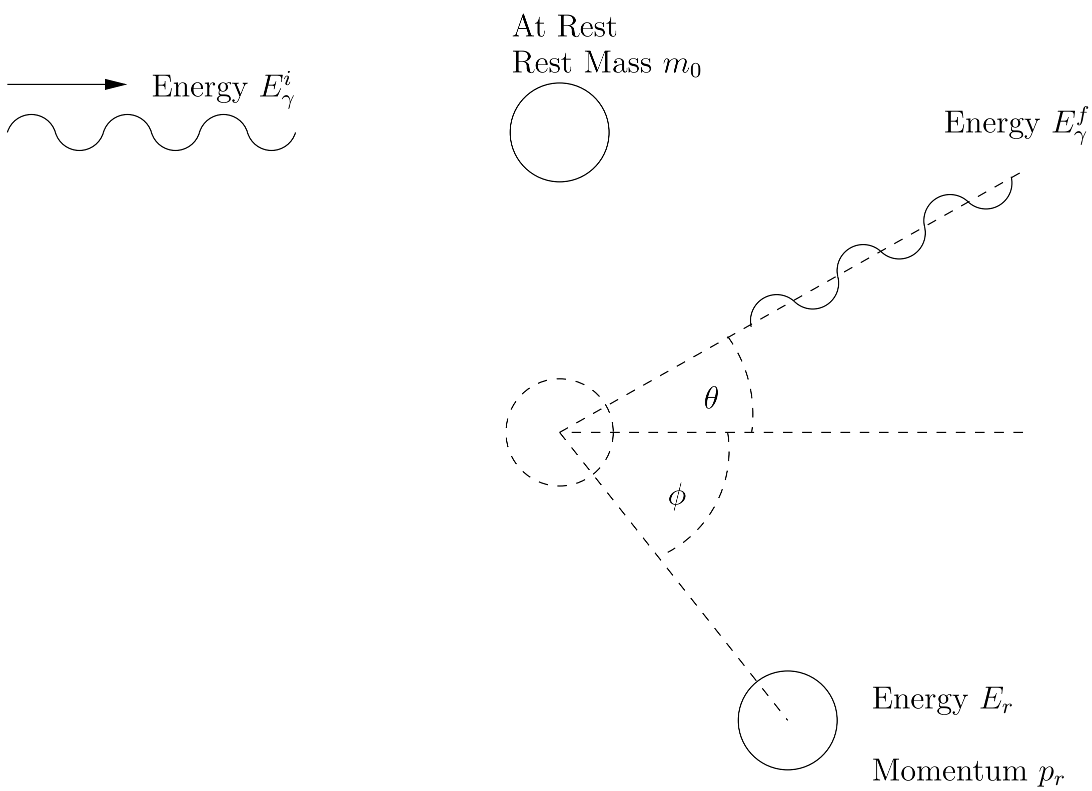

Figure 1: Compton scattering. An incident photon having energy E_γ^i strikes a stationary particle of rest mass m_0. The particle recoils with total energy E_r and momentum of magnitude p_r, and the photon moves off with energy E_γ^f. As a consequence, energy is transferred from the incident photon to the recoiling particle, and the direction of the photon is altered.

Let us use conservation of energy and momentum to analyze Compton scattering. First, momentum. The interaction occurs entirely within the plane formed by the incident direction of the photon and the scattering directions of the recoiling particle and photon, otherwise it would be impossible to conserve momentum. So there are two other momentum components that are conserved, that parallel to the direction of incidence of the photon and that perpendicular to it. Conservation of momentum parallel to the incident photon first:

$$p_\gamma^i = p_r \cos\phi + p_\gamma^f \cos\theta, \tag{4}$$

where p_γ^i is the momentum of the incident photon. Multiplying by c we obtain

$$p_\gamma^i c = p_r c \cos\phi + p_\gamma^f c \cos\theta. \tag{5}$$

Using Equation 3 to substitute for $p_\gamma^i c$ and $p_\gamma^f c$ (we cannot do this for the particle because it has a mass), we obtain

$$E_\gamma^i = p_r c \cos\phi + E_\gamma^f \cos\theta. \tag{6}$$

Now what we are heading for here is an equation for the energy shift of the photon in terms of its scattering angle θ. We would therefore like to eliminate ϕ, so let us isolate the term that has a ϕ in it. Moving the rightmost term to the left of the = sign, we obtain

$$E_\gamma^i - E_\gamma^f \cos\theta = p_r c \cos\phi. \tag{7}$$

So much for the first momentum component. Notice, though, that the momentum component in the direction perpendicular to the direction of the incident photon is also conserved. In this direction, there is zero momentum component before the collision, because neither particle has any component of its velocity in that direction. Therefore the sum of the momentum components in the vertical direction on the figure after the collision must also be zero. So we write

$$p_\gamma^f c \sin\theta = p_r c \sin\phi. \tag{8}$$

We have included a c in anticipation of the next step, which is to replace photon momentum with energy,

$$E_\gamma^f \sin\theta = p_r c \sin\phi. \tag{9}$$

Now we can eliminate ϕ between Equations 7 and 9 by squaring both equations. Squaring Equation 7 we obtain

$$(E_\gamma^i - E_\gamma^f \cos\theta)^2 = p_r^2 c^2 \cos^2\phi. \tag{10}$$

Squaring Equation 9 we get

$$(E_\gamma^f)^2 \sin^2\theta = p_r^2 c^2 \sin^2\phi. \tag{11}$$

Adding these two equations together we can eliminate the unwanted angle ϕ.

$$(E_\gamma^i - E_\gamma^f \cos\theta)^2 + (E_\gamma^f)^2 \sin^2\theta = p_r^2 c^2 \left(\cos^2\phi + \sin^2\phi\right) = p_r^2 c^2. \tag{12}$$

Continuing to simplify this equation, we multiply out the brackets on the left hand side.

$$(E_\gamma^i)^2 - 2E_\gamma^i E_\gamma^f \cos\theta + (E_\gamma^f)^2 \cos^2\theta + (E_\gamma^f)^2 \sin^2\theta = p_r^2 c^2, \quad (13)$$

and therefore

$$(E_\gamma^i)^2 - 2E_\gamma^i E_\gamma^f \cos\theta + (E_\gamma^f)^2 = p_r^2 c^2. \quad (14)$$

Getting to this stage has used up the laws of conservation of momentum components. Next we apply conservation of energy.

$$E_\gamma^i + m_0 c^2 = E_\gamma^f + E_r. \quad (15)$$

Here we are moving towards eliminating the energy E_r and momentum p_r of the recoiling particle. So we isolate E_r by moving E_γ^f to the other side of the equals sign.

$$E_\gamma^i + m_0 c^2 - E_\gamma^f = E_r. \quad (16)$$

Next we square this equation, because we notice that E_r^2 could be added to the $p_r^2 c^2$ on the right hand side of Equation 14 to eliminate both of them. Squaring the left hand side will give us six terms, but that's life. Let's hope a lot of them cancel later.

$$(E_\gamma^i + m_0 c^2 - E_\gamma^f)^2 = E_r^2, \quad (17)$$

or

$$(E_\gamma^i)^2 + m^2 c^4 + (E_\gamma^f)^2 + 2E_\gamma^i mc^2 - 2E_\gamma^i E_\gamma^f - 2E_\gamma^f mc^2 = E_r^2. \quad (18)$$

Now we subtract Equation 14 from Equation 18. The right hand sides subtract to give $E_R^2 - p_r^2 c^2 = m_0^2 c^4$. This means we have eliminated E_r and p_r leaving the following,

$$m_0^2 c^4 + 2E_\gamma^i mc^2 - 2E_\gamma^i E_\gamma^f (1 - \cos\theta) - 2E_\gamma^f mc^2 = m^2 c^4. \quad (19)$$

After cancelling the $m_0^2 c^4$ terms on the left and the right, cancelling the factors of 2 and rearranging, we are left with

$$\frac{(E_\gamma^i - E_\gamma^f)}{E_\gamma^i E_\gamma^f} = \frac{1 - \cos\theta}{m_0 c^2}. \quad (20)$$

This is a lot prettier than some of the long winded equations from earlier, but it's still not as compact as it can be in wavelength terms. The wavelength λ of a photon of energy E is

$\lambda = hc/E$. Denoting the final and initial photon wavelengths by λ^f and λ^i, we write down the wavelength difference,

$$\begin{aligned} \lambda^f - \lambda^i &= \tfrac{hc}{E_\gamma^f} - \tfrac{hc}{E_\gamma^i} \\ &= \tfrac{hc\left(E_\gamma^i - E_\gamma^f\right)}{E_\gamma^i E_\gamma^f} \end{aligned} \tag{21}$$

Substituting in Equation 20 leads us to a simple form of the Compton scattering formula relating the wavelength shift in the photon to the scattering angle θ

$$\lambda^f - \lambda^i = \frac{hc(1 - \cos\theta)}{m_0 c^2}. \tag{22}$$

4 Compton scattering in detectors

The Compton scattering result is one of the most important results that can be easily derived using ordinary conservation of energy and momentum in relativity. The physics consequences of this result are far reaching because of the nature of many detectors in particle physics. Very common types of detectors for radiation in the form of γ–rays make use of materials that rely on the absorption of gamma rays in some solid detector element, perhaps a crystal of germanium, or sodium iodide.

What you want to happen is that all the energy of the γ–ray is absorbed in the material, which then re-emits that energy in some detectable form, perhaps a flash of light or some ionisation electrons, both of which can be detected in suitable instruments (a photomultiplier tube for the flash of light, some electrodes and charge amplifier electronics for the ionisation electrons). In which case, all the energy of the incoming particle is detected, and if the incoming γ–rays were mono-energetic, you see a nice narrow spike in the energy spectrum of the detected light or ionisation charge.

In reality, various affects can spoil the party. One such affect is compton scattering. Your gamma ray scatters off some electron either in the source or in the detector, imparting a portion of its energy to the electron. This electron then itself deposits energy in the detector, so some of the scintillation or ionisation pulses detected by your instrumentation will have an energy characteristic of the scattered electron, not that of the incident γ–ray.

Notice, however, that there is a gap between the maximum energy of these scattered electrons and the incident energy of the γ–ray that produces them.

Start with Equation 20, and note that the maximum energy transfer to the electron will be when $\theta = 180°$, so that $1-\cos\theta = 2$. Also write $E_\gamma^i - E_\gamma^f = \Delta$. Then the equation becomes

$$\frac{\Delta}{E_\gamma^i(E_\gamma^i - \Delta)} = \frac{2}{m_0c^2}. \tag{23}$$

Solving for Δ we obtain

$$\Delta = \frac{(E_\gamma^i)^2}{E_\gamma^i + \frac{mc^2}{2}}. \tag{24}$$

This gap means even the most energetic Compton scattered electrons cannot match the energy of the incident γ–ray. There will always be an energy gap between signals corresponding to complete absorption of the incident γ–ray by the detector and signals corresponding to the most energetic compton scattered electrons. The typical spectrum of detector signals consists of a sharp full absorption peak at high energy with a broader distribution having an abrupt shoulder falling off at energies slightly below this peak. This shoulder is called the Compton edge, and is seen in a large variety of spectra from radiation detectors in many fields of physics. An example of a gamma-ray spectrum from a sintered AmBe source (a rich source of gamma rays and neutrons) is shown in Figure 2. There are two full absorption peaks visible, one very narrow one at channel 2200 or thereabouts, and the other slightly wider one at 4360. Notice that below each of these peaks is a Compton edge, corresponding the maximum energy compton scattered electrons from gamma rays in these two narrow peaks.

The other features labelled escape peaks are due to gamma rays inducing electron-positron pair production in the detector, where either one of the pair (single-escape peak) or both of them (double-escape peak) leaves the detector without depositing its energy.

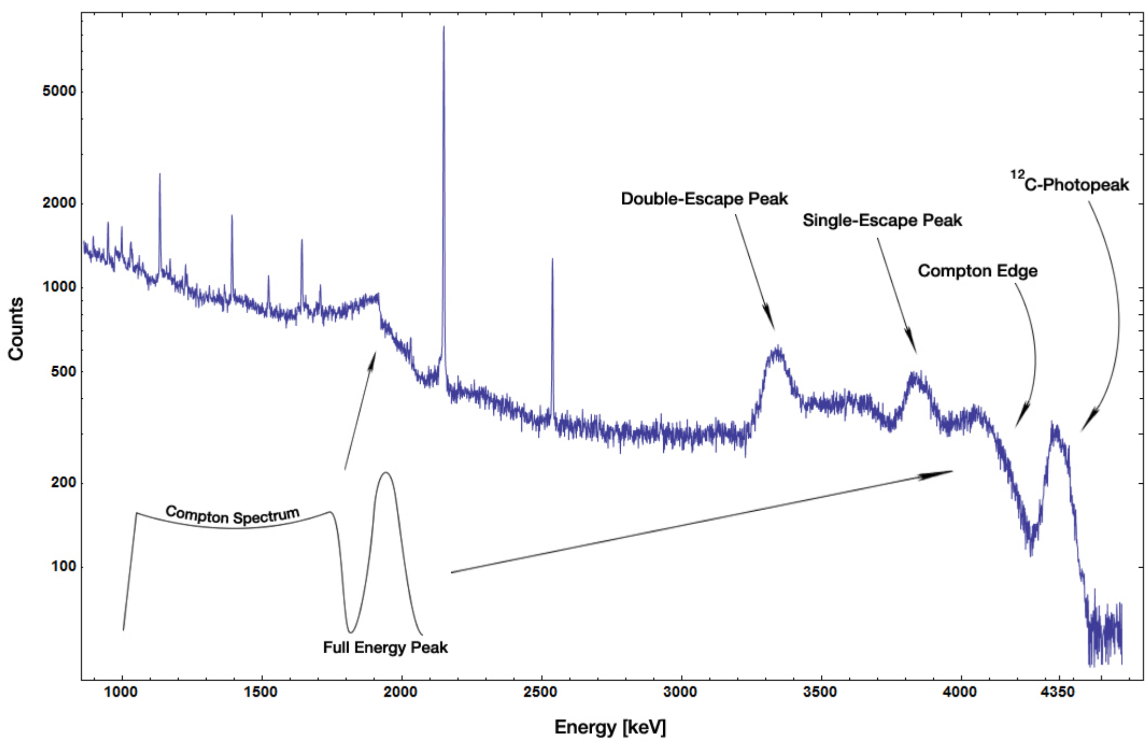

Figure 2: The energy spectrum of gamma rays from an AmBe source, as measured by a gamma ray spectrometer. Taken from `http://upload.wikimedia.org/wikipedia/commons/f/f2/Am-Be-SourceSpectrum.jpg`