Homework, PHY206

1. An ultra high energy cosmic ray proton with a lab frame energy of 10^{20} eV strikes a cosmic ray background photon having an equivalent temperature of 4.5 degrees kelvin. The proton rest mass is $938\,\mathrm{MeV/c^2}$.
(a). What is the energy of the cosmic ray photon in eV?
(b). What is the gamma factor, γ, for the proton?
(c). In the rest frame of the proton, what is the energy of the approaching cosmic ray photon, in eV?

2. A honda civic travelling at high speed in a car park collides head-on with a stationary parked volvo. The volvo recoils at a speed of $1\,\mathrm{m\,s^{-1}}$.
(a) What are the SI units for q^2? Give in terms of kilograms, metres and seconds.
(b) If the mass of the volvo is 1100kg, what is q^2 for the collision, in SI units?
(c) What is q^2 for the collision in $(eV/c)^2$? This will be a huge number.

3. Re-read the notes from the lecture on Compton scattering - lecture 3.
(a) From Equations 7 and 8 in the notes, eliminate the terms in θ to obtain

$$p_r^2c^2 = {E_\gamma^f}^2 - {E_\gamma^i}^2 + 2E_\gamma^i p_r c\cos\phi.$$

(b) Now combine this result with Equation 17 to obtain

$$E_\gamma^i p_r c\cos\phi = {E_\gamma^i}^2 - E_\gamma^i E_\gamma^f + E_\gamma^i m_0c^2 - E_\gamma^f m_0c^2.$$

(c) Now rearrange and eliminate p_rc using one of the conservation equations to show that

$$\cos\phi = \frac{(E_\gamma^i + m_0c^2)(E_\gamma^i - E_\gamma^f)}{E_\gamma^i\sqrt{(E_\gamma^i - E_\gamma^f + m_0c^2)^2 - m_0^2c^4}}.$$

(d) Show using the result from (c) that the component of the recoil velocity of the nucleus parallel to the direction of incidence of the photon is always positive.

1 Solutions to PHY206 relativity homework problems

Solution to problem 1.
(a) A gas of temperature 4.5 K has an average of $\mathrm{k_B T} = 1.38\times10^{-23}\,[\mathrm{J\,K^{-1}}]\times 4.5\,[\mathrm{K}] = 6.2\times10^{-23}$ J, or 3.9×10^{-4} eV.

(b) $\gamma = E/(M_p c^2) = 10^{20}\,[\mathrm{eV}]/938 \times 10^6\,[\mathrm{eV}] = 1.07 \times 10^{11}$. Because of the error in the original problem sheet where I quoted the proton mass as incorrectly as $938\,\mathrm{GeV/c^2}$, I will also allow the answer that this assumed mass would have given for γ, which is 1.07×10^8.
(c) The proton is in the highly relativistic regime, so that transforming to its rest frame we set $\beta = 1$, and use the γ found in Section (b). We assume the photon is moving to the left towards the incoming proton, though I did not explicitly state this in the question, so I will allow other possibilities in your answers. If this is so, its momentum is $p = -E/c$, with the minus sign indicating the direction of motion. We use the Lorentz transformation formula for energy,

$$\frac{E'}{c} = \gamma \left(\frac{E}{c} - \beta p \right) .$$

Substituting in for E and p and setting $\beta = 1$, I obtain $E' = \gamma(E - \beta(-E)) = 2\gamma E$. Therefore $\mathrm{E}' = 2 \times 1.07 \times 10^{11} \times 3.9 \times 10^{-4}\,[\mathrm{eV}] = 83\,\mathrm{MeV}$.

For solutions to problems 2 and 3, see hand–written sheets following this one.

PHY 206 Relativity
Homework – Solutions

1. a) $E_\gamma = k_B T$

$$= 1.38\times10^{-23}\ \mathrm{J\,K^{-1}} \times 4.5\ \mathrm{K}$$
$$= 6.21\times10^{-23}\ \mathrm{J}$$
$$= 3.9\times10^{-4}\ \mathrm{eV}$$

b) $$\gamma = \frac{E_p}{m_p c^2} = \frac{10^{20}\ \mathrm{eV}}{938\times10^{6}\ \mathrm{eV}} = 1.07\times10^{11}$$

c) Lorentz boost to rest frame of proton.

energy E_p energy E_γ

momentum $\frac{E_p}{c}$ momentum $-\frac{E_\gamma}{c}$

$$\frac{E_\gamma'}{c} = \gamma\left(\frac{E_\gamma}{c} - \beta\left(-\frac{E_\gamma}{c}\right)\right)$$

but $\beta \simeq 1$ for $\gamma = 10^{11}$ – highly relativistic,

$$E_\gamma' = 1.07\times10^{11}\,(2E_\gamma)$$
$$E_\gamma' = 83.2\ \mathrm{MeV}.$$

2. $Q^2 = 2M\nu$ M: volvo (target) mass

ν: energy transfer to volvo

$$\nu = \tfrac{1}{2}Mv^2$$
$$= \tfrac{1}{2}\times 1100\,[\mathrm{kg}]\times 1^2\,[\mathrm{m^2 s^{-2}}]$$
$$\nu = 550\ \mathrm{kg\,m^2 s^{-2}}$$

$$Q^2 = 2\times1100\ \mathrm{kg}\times 550\ \mathrm{kg\,m^2 s^{-2}}$$
$$= 1.21\times10^{6}\ (\mathrm{kg\,m\,s^{-1}})^2$$

units answer part a

number answers part b

$$\Rightarrow \boxed{Q^2 = 4.3\times10^{60}\,(\mathrm{eV}/c)^2}$$

Q is a momentum, $Q = \sqrt{1.21\times10^{6}} = 1100\ \mathrm{kg\,m\,s^{-1}}$

Energy is $Qc = 1100\times3\times10^{8}\ \mathrm{kg\,m^2 s^{-2}} = 3.30\times10^{11}\ \mathrm{J} = 2.1\times10^{30}\ \mathrm{eV}.$

H/W problem 3

a) $E_\gamma^i = p_r c \cos\varphi + E_\gamma^f \cos\theta$ ← Equation 7

$E_\gamma^f \cos\theta = E_\gamma^i - p_r c \cos\varphi$

$E_\gamma^{f\,2} \cos^2\theta = E_\gamma^{i\,2} - 2E_\gamma^i p_r c \cos\varphi + p_r^2 c^2 \cos^2\varphi$

$E_\gamma^{f\,2} \sin^2\theta = p_r^2 c^2 \sin^2\varphi$ ← Square of Equation 8

+

$E_\gamma^{f\,2} = E_\gamma^{i\,2} - 2E_\gamma^i p_r c \cos\varphi + p_r^2 c^2$

$p_r^2 c^2 = E_\gamma^{f\,2} - E_\gamma^{i\,2} + 2E_\gamma^i p_r c \cos\varphi$ * result of part 3a

b) $E_r^2 = (E_\gamma^i + m_0 c^2 - E_\gamma^f)^2$

$\underbrace{E_r^2 - p_r^2 c^2}_{} = (E_\gamma^i + m_0 c^2 - E_\gamma^f)^2 - E_\gamma^{f\,2} + E_\gamma^{i\,2} - 2E_\gamma^i p_r c \cos\varphi$

$\Rightarrow \cancel{m_0^2 c^4} = E_\gamma^{i\,2} + \cancel{m_0^2 c^4} + \cancel{E_\gamma^{f\,2}} + 2E_\gamma^i m_0 c^2 - 2E_\gamma^i E_\gamma^f - 2m_0 c^2 E_\gamma^f - \cancel{E_\gamma^{f\,2}} + E_\gamma^{i\,2} - 2E_\gamma^i p_r c \cos\varphi$

$0 = \cancel{2}E_\gamma^{i\,2} + \cancel{2}E_\gamma^i m_0 c^2 - \cancel{2}E_\gamma^i E_\gamma^f - \cancel{2}m_0 c^2 E_\gamma^f - \cancel{2}E_\gamma^i p_r c \cos\varphi$

$E_\gamma^i p_r c \cos\varphi = E_\gamma^{i\,2} - E_\gamma^i E_\gamma^f + E_\gamma^i m_0 c^2 - E_\gamma^f m_0 c^2$ * result of 3b

c.) $p_r c = \sqrt{E_r^2 - m_0^2 c^4}$

$= \sqrt{(E_\gamma^i - E_\gamma^f + m_0 c^2)^2 - m_0^2 c^4}$.

$E_\gamma^i p_r c \cos\varphi = E_\gamma^i (E_\gamma^i - E_\gamma^f) + m_0 c^2 (E_\gamma^i - E_\gamma^f)$

$= (E_\gamma^i + m_0 c^2)(E_\gamma^i - E_\gamma^f)$

combine, $\cos\varphi = \dfrac{(E_\gamma^i + m_0 c^2)(E_\gamma^i - E_\gamma^f)}{E_\gamma^i \sqrt{(E_\gamma^i - E_\gamma^f + m_0 c^2)^2 - m_0^2 c^4}}$

d.) $\cos\varphi$ is always positive, so φ is always between $-90°$ and $+90°$. Therefore the recoil direction of the scattered electron never has a component pointing back towards the incident gamma.