1^{st} Problem Class, PHY206

1. A Δ^0 baryon, rest mass $M_\Delta = 1116\,\mathrm{MeV/c^2}$, decays at rest in the lab into a proton, rest mass $M_p = 938\,\mathrm{MeV/c^2}$ and a π^- meson, rest mass $M_\pi = 135\,\mathrm{MeV/c^2}$:

$$\Delta^0 \to p\,\pi^-.$$

In the lab frame:

(a) [1] Argue that the magnitudes of the momenta of the proton and the π^- meson after the decay are equal. Denote either by p.
(b) [1] Express the total energy E_p of the proton after the decay in terms of p and the proton rest mass.
(c) [1] Express p_π in terms of the total energy E_π of the π^- meson after the decay and the π^- meson rest mass.
(e) [2] Show using conservation of total energy that

$$\sqrt{p^2c^2 + M_p^2c^4} + \sqrt{p^2c^2 + M_\pi^2c^4} = M_\Delta c^2.$$

(f) [4] Solve for p, showing that

$$pc = c^2\sqrt{\frac{(M_\Delta^2 - M_p^2 - M_\pi^2)^2 - 4M_p^2M_\pi^2}{4M_\Delta^2}}.$$

(g) [1] Substitute in the masses and calculate a value for pc in MeV/c. Be extremely careful to square all the quantities correctly and to perform calculations in the correct order.

Total of 10 marks, with division of marks awarded shown above.

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(a) In the lab frame, total momentum before decay was zero, so total momentum after decay must also be zero. With two products, their momenta must be equal and opposite, hence their momentum magnitudes are the same, p.
(b) Use the formula $E^2 = (pc)^2 + (m_0c^2)^2$ to obtain $E_p = \sqrt{p^2c^2 + M_p^2c^4}$.
(c) Similar to (b), $E_\pi = \sqrt{p^2c^2 + M_\pi^2c^4}$.
(d) Using conservation of energy, total energy before is the rest energy of the Δ_0, which is $M_\Delta c^2$. Therefore $E_p + E_\pi = M_\Delta c^2$. Substituting in the answers to parts (b) and (c) we get the required result.
(e) First step is to square the result of (d), obtaining

$$2p^2c^2 + M_p^2c^4 + M_\pi^2c^4 + 2\sqrt{(p^2c^2 + M_p^2c^4)(p^2c^2 + M_\pi^2c^4)} = M_\Delta^2c^4.$$

Next step is to isolate the square root on the left hand side, yielding

$$2\sqrt{(p^2c^2 + M_p^2c^4)(p^2c^2 + M_\pi^2c^4)} = M_\Delta^2c^4 - M_p^2c^4 - M_\pi^2c^4 - 2p^2c^2.$$

Next step is to square again.

$$\begin{aligned}&4p^4c^4 + 4p^2c^2M_\pi^2c^4 + 4p^2c^2M_p^2c^4 + 4M_p^2c^4M_\pi^2c^4 = \\ &(M_\Delta^2c^4 - M_p^2c^4 - M_\pi^2c^4)^2 - 4p^2c^2(M_\Delta^2c^4 - M_p^2c^4 - M_\pi^2c^4) + 4p^4c^4.\end{aligned}$$

The $4p^4c^4$ terms now cancel, as do terms in $4p^2c^2M_\pi^2c^4$ and $4p^2c^2M_p^2c^4$. When you've done cancelling, you can rearrange so that the remaining term having a p^2 in it, $4p^2c^2M_\Delta^2c^4$ is isolated, yielding

$$4p^2c^2M_\Delta^2c^4 = (M_\Delta^2c^4 - M_p^2c^4 - M_\pi^2c^4)^2 - 4M_p^2c^4M_\pi^2c^4.$$

Rearranging this formula making the subject p^2c^2 and taking the square root gives the desired result.
(g) The correct answer is, I believe, 107 MeV/c. This is found by substituting in the values of the masses in MeV/c for the Δ, proton, and π^- meson, from the question.