Lecture 8 - Relativistic energy and momentum — 2

E. Daw

April 4, 2011

1 Review of lecture 7

Last time we worked out an expression for the conserved energy E associated with a moving particle of rest mass m_0. It was:

$$E = \gamma m_0 c^2. \tag{1}$$

This result was guessed, and the guess then checked. The guess involved studying the decay of a particle of rest mass M_0 into a lighter particle and a photon. A portion m_0 of its mass is used to create a photon having energy $E_\gamma = m_0 c^2$ and momentum $p_\gamma = E_\gamma / c = -m_0 c$. We guessed that energy and momentum are still conserved in relativistic problems, so that by conservation of momentum the particle produced in the decay has energy $E_f = (M_0 - m_0)c^2$ and momentum $p_f = +m_0 c$.

We next considered what happens in the rest frame of the decay particle. In this frame, the initial particle was moving before it decayed with a velocity $-\beta c$. In this frame, the decaying particle is moving, so that the total energy available is by our guess $\gamma M_0 c^2$, where $\gamma = 1/\sqrt{1-\beta^2}$. Consider an observer at the origin, with the decay photon moving towards her. Because it is moving towards her, it will be blue shifted, so its energy will be

$$E'_\gamma = E_\gamma \sqrt{\frac{1+\beta}{1-\beta}} = m_0 c^2 \sqrt{\frac{1+\beta}{1-\beta}}. \tag{2}$$

Therefore using conservation of energy in this frame, the total energy available for the final state particle is the total energy of the initial particle minus the energy of the photon, or,

$$E'_f = E'_i - E'_\gamma = \gamma M_0 c^2 - m_0 c^2 \sqrt{\frac{1+\beta}{1-\beta}}. \tag{3}$$

Because it is a photon moving to the left towards the origin, its momentum is $-E, /c$, which is

$$p'_\gamma = \frac{-E'_\gamma}{c} = -m_0 c \sqrt{\frac{1+\beta}{1-\beta}}. \tag{4}$$

We next worked out E'_γ another way. We guessed that the energy and the three components of the momentum of a particle can be written together as a four component object, $(E/c, p^x, p^y, p^z)$, analogous to the four components of a spacetime point (ct, x, y, z). Furthermore, perhaps these four component objects transform the same way between observers moving at different velocities. We already know how to transform the components of a spacetime point, so we wrote down the same transformations for energy and momentum. For two observers whose relative velocity is along the x–axis, we get.

$$\begin{aligned} E'/c &= \gamma(E/c - \beta p^x) \\ p'^x &= \gamma(p^x - \beta E/c) \\ p'^y &= p^y \\ p'^z &= p^z \end{aligned} \tag{5}$$

To test this rather radical, but potentially very useful idea, we apply to the current problem. The energy and momentum of the final state particle in the rest frame of the initial particle that decays are $E_f = (M_0 - m_0)c^2$ and momentum $p_f = +m_0 c$. Substitute these into the energy equation above to get the final state energy in the primed frame.

$$
\begin{aligned}
E'_f/c &= \gamma(E_f/c - \beta m_0 c) \\
&= \gamma(M_0 c - m_0 c - \beta m_0 c) \\
&= \gamma(M_0 c - (1+\beta) m_0 c) \\
&= \gamma M_0 c - \frac{(1+\beta)}{\sqrt{1-\beta^2}} m_0 c \\
&= \gamma M_0 c - \sqrt{\frac{(1+\beta)^2}{(1+\beta)(1-\beta)}} m_0 c \\
E'_f &= \gamma M_0 c^2 - \sqrt{\frac{1+\beta}{1-\beta}} m_0 c^2.
\end{aligned} \tag{6}
$$

This is as far as we got last time. The agreement between these two methods, and in particular the agreement between the use of Lorentz transformations on the four components of the energy-momentum four vector and the use of the doppler shift, confirms the picture where the total energy of a moving particle is $\gamma M_0 c^2$ and the four compoents $(E/c, p^x, p^y, p^z)$ transform under changes of observer exactly like the four components of a space-time event, (ct, x, y, z).

2 Momentum

Let us now use these same transformations to figure out a formula for the relativistic momentum of a moving particle. The easiest way is to consider the decaying particle of mass M_0 above. It's energy in the unprimed frame, its own rest frame, is $M_0 c^2$, and its momentum is zero. Using the second of Equations 5, we figure out the momentum in the primed frame, where the particle is moving along the x–axis to the left at velocity $v = -\beta c$.

$$
p'^x = \gamma(p^x - \beta E'/c) = -\beta\gamma M_0 c. \tag{7}
$$

So the momentum of a particle of rest mass m_0 and velocity $\vec{v}$ is

$$
\vec{p} = \gamma m_0 \vec{v} \tag{8}
$$

3 Examples

A very rare but occasionally observed decay of the muon is into an electron and a photon. If a muon decays at rest by this mode, what is the energy of each decaying particle? And what is the magnitude of the electron momentum? The rest masses of the muon and electron are $(106\,\mathrm{MeV})/\mathrm{c}^2$ and $(0.511\,\mathrm{MeV})/\mathrm{c}^2$, respectively.

4 Inertial mass

So far I have been very careful to use a 0 subscript whenever I write down a rest mass. The reason I have done this is that there is another distinct mass that is sometimes used, called the inertial mass. Most of you have been taught about inertial masses to some extent at A level, though some of you may not have realised that inertial mass and rest mass are different, and have different properties. In this section I will address the inertial mass, and give some perspectives on the results that lead from this idea, with commentary. I will then not use inertial masses at all for the remainder of the course, and will scrupulously put 0 subscripts by all the rest masses that appear in our lectures, problems, and discussions, since having two varieties of mass will probably lead to confusion, and the rest mass is easily the predominant one in modern texts and courses at other Universities.

The inertial mass is the ratio of the momentum of a body to its velocity. In non-relativistic physics this is an intuitive idea. Take a football and kick it so that it acquires a velocity of $5\,\mathrm{m/s}$. It's quite easy to catch it and stop it moving. Take a lead ball of the same size and give it a velocity of $5\,\mathrm{m/s}$. Try stopping it and you will conclude that it has a higher mass. In pre-relativity physics, this is no problem because rest mass and inertial mass are equivalent. For the football, its momentum is $p = mv$, where m is just its mass, the mass that it would have at rest, the mass that you would get if you weighed it. Divide momentum by velocity and you get the inertial mass, which is also m. There is no problem.

In relativity, the ratio of momentum to velocity is

$$\frac{p}{v} = \gamma m_0, \tag{9}$$

where m_0 is the rest mass. The inertial mass, defined as p/v, leads to

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{10}$$

So the inertial mass of a body of rest mass m_0 grows as the velocity of the particle increases, and becomes infinite as v tends to c. By contrast, its rest mass m_0 is independent of it's velocity, since we can only measure its rest mass when it is at rest, and the value we will get is m_0. The rest mass is an example of an invariant, a quantity that does not change with the velocity of the object with respect to the observer.

Some of you will remember being taught at A level that 'as an object gets faster, its mass increases', or something like that. I have put this in quotes because as it stands it is an imprecise and therefore confusing statement. It is true that as an object gets faster, its inertial mass increases. This is the consequence of Equation 10. However, it is also true that the rest mass of a particle is independent of its velocity - a particle has the same rest mass whether it is moving at 0.99 of the speed of light or sitting stationary on the desk next to you.

What about energy? From equation 1 you can express the total energy E in terms of the inertial mass m, as follows:

$$E = (\gamma m_0)c^2 = mc^2. \tag{11}$$

In terms of inertial mass E is always equal to inertial mass times c^2, no matter what the velocity of the particle. In this way of considering the particle, the increase in energy when the particle starts moving is a consequence of the increase in inertial mass.

One can take the concept of inertial mass further, and discuss photons. The rest mass of a photon is zero. However, what about the inertial mass? We know that for a photon of energy E, the momentum has magnitude

$$p = \frac{E}{c}. \tag{12}$$

The ratio of momentum to velocity is therefore

$$\frac{p}{c} = \frac{E}{c^2} \tag{13}$$

Following our definition of inertial mass to the ultimate, the ratio of the photon energy E to c^2 is the inertial mass of a photon. So the inertial mass of a photon is proportional to its energy, and is certainly not zero. So in the case of photons, the situation is extreme. Their rest mass is zero, their inertial mass is not. Clearly the use of both rest and inertial mass in our course is going to be confusing.

However, there is a way out of this dilemma, and that is to realize that it is always true, both for particles having non-zero rest mass and photons having zero rest mass, that their total energy E is proportional to their inertial mass m. Since total energy is conserved, it follows that total inertial mass is conserved too. Energy and inertial mass differ only by a constant factor, c^2. Many of you will have heard of Occam's razor, the philosophical principal that can be stated (look it up in Wikipedia):

> Entitites must not be multiplied beyond necessity.

Here we have a classic case of entities being multiplied beyond necessity. The energy and the inertial mass are, up to a constant, the same thing. Therefore, abandon use of one of them completely and you achieve simplification without losing anything. Since we are all used to energy, let us therefore abandon any future reference to inertial mass, m, and use only rest masses m_0, which I will continue to write scrupulously with a 0 subscript to make absolutely sure that you know what I mean.

I have been careful to provide a definition of inertial mass as the ratio of momentum to velocity, so I should carefully provide a definition for rest mass as well. I think this is the nicest one. Take your particle at rest and convert this particle by some means into photons. Collect all the energy of these photons up in some way, measure the total energy of all the photons,

and divide by c^2, and you have determined the rest mass of the particle. What about the photon? Imagine a stream of photons being emitted by some source. They are already photons so all you have to do to measure their rest mass is to boost yourself to as close to the speed of light as you can. To an observer in the rest frame of the source, you can approach the speed of the photons if you have enough energy. So you boost yourself up to a velocity close to that of light, the velocity of the photons. But consider - this means that your frame of reference is one in which the source of the photon was receding from you at close to the speed of light. In this frame, the photon's energy is highly redshifted. In the limit where your velocity approaches the speed of light, and 'catch up' with the photon, it's energy is redshifted to zero, and therefore it's rest mass is also zero.

Finally, note that it is not correct to substitute the inertial mass into some other expressions that you are familiar with from classical physics. For example you cannot produce a relativistic analogy to kinetic energy by taking the classical formula $mv^2/2$ and placing the inertial mass in place of the m. The correct relativistic kinetic energy is the difference between the total energy of the particle and its rest energy. So the kinetic energy T of a particle is

$$T = \gamma m_) c^2 - m_0 c^2 = (\gamma - 1) m_0 c^2 \tag{14}$$

Unlike the total energy, the kinetic energy is not a conserved quantity, since the total energy also includes the rest energy, essentially a form of potential energy, the energy locked away in the rest mass of the particle.

5 An introduction to invariants

I have occasionally referred to some quantities as being invariant, meaning that different inertial observers will agree upon their value. For example, speed of light in a vacuum, invariant due to Einsteins second postulate of special relativity. Another invariant that I have just discussed is the rest mass. A third one is the proper time interval between two events.

Other quantities are not invariants. For example, the set of

quantities $(E/c, p^x, p^y, p^z)$

$$\begin{aligned} \frac{E'}{c} &= \gamma\left(\frac{E}{c} - \beta p^x\right), \\ p^{x\prime} &= \gamma\left(p^x - \beta\frac{E}{c}\right), \\ p^{y\prime} &= p^y, \\ p^{z\prime} &= p^z. \end{aligned} \tag{15}$$

Let us now see if we can form combinations of these coordinates that are invariant. Consider first a particle whose momentum is directed along the x–axis with respect to an observer. The energy of the particle is $E = \gamma m_0 c^2$, and its momentum is $p = \gamma m_0 v$. Let us form the combination $E^2 - p^2c^2$. Substituting in for E and p we obtain

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m_0^2 c^4 - \gamma^2 m_0^2 v^2 c^2 \\ &= \gamma^2 m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m_0^2 c^4, \end{aligned} \tag{16}$$

where the last line follows because γ^2 is equal to $1/(1 - v^2/c^2)$. The right hand side is a combination of the rest mass and c, both invariant quantities. The left hand side contains no factors of β or γ, and therefore this combination of relativistic total energy and momentum is independent of the velocity of the observer with respect to the particle. Suppose we have two observers, both moving along the x–axis with respect to the particle, but having different velocities relative to it. Then we can use the invariance of this combination to deduce that

$$E^2 - p^2c^2 = E'^2 - p'^2c^2 = m_0^2c^4. \tag{17}$$

The latter equality reinforces the idea that this quantity is invariant; for the special observer with respect to whom the particle is at rest, the energy is equal to the rest energy, so that $E_R = m_0c^2$, and the momentum is zero, $p_R = 0$, so that $E_R^2 - p_R^2c^2 = (m_0c^2 - 0)^2 = m_0^2c^4$.