Lecture 7 - Relativistic energy and momentum – 1

E. Daw

April 4, 2011

1 Review of relativistic doppler shift

Last time we figured out the relativistic generalisation of the classical doppler shift of light emitted by a moving source. For a source that is moving away from the observer at a velocity $v = \beta c$ parallel to the straight line joining source and observer, the wavelength λ_{ob} observed by the observer with respect to whom the source is receding is related to the wavelength λ_{em} in the rest frame of the source by

$$\lambda_{\mathrm{ob}} = \lambda_{\mathrm{em}} \sqrt{\frac{1+\beta}{1-\beta}}. \tag{1}$$

This leads directly to the Doppler shift z of the light, which is greater than zero for the source receding from the observer and less than zero for the source approaching the observer.

$$z = \frac{\lambda_{\mathrm{ob}} - \lambda_{\mathrm{em}}}{\lambda_{\mathrm{em}}} = \sqrt{\frac{1+\beta}{1-\beta}} - 1. \tag{2}$$

2 Energy associated with a mass at rest

In this section we will derive the most famous result of Einstein's special theory. The derivation is reproduced from French's book, and like the light clock derivation of time dilation it is not rigorous, but does give a nice intuitive feel for the origin of the result. Before we set up the proof, we need to recall a result which comes from classical electromagnetism, the relationship between the energy and momentum of light. If a light beam carries momentum of magnitude p and energy E, then these are related by

$$E = cp. \tag{3}$$

You may wonder how this very relativistic looking result arises from classical electromagnetism! In fact, it turns out that the equations of classical electromagnetism are consistent with special relativity, even though they pre-date it by some forty years. In fact, Einstein and others came up with the Lorentz transformations by considering electromagnetism; Einstein's famous paper in which special relativity is discussed is called 'The Electrodynamics of Moving Bodies'. We will need this result in the derivation to come.

We consider a closed box free to move along the x–axis. The box is of mass M and initially at rest. This box contains a hollow cavity of length L. At some time the box emits a photon from the left hand end of the internal cavity, which travels down the cavity and is re-absorbed at the right hand end. Figure 1 is a schematic of this apparatus.

Consider the photon to have energy E_γ. Therefore, its momentum is $p_\gamma = E_\gamma/c$. To conserve momentum, the momentum of the box during the photon's flight must be $-E_\gamma/c$. Since the box is moving with $v \ll c$, we can find its velocity by writing $Mv = E_\gamma/c$. Therefore the velocity of the box is $v = E_\gamma/(Mc)$. Now, if the box is moving very slowly, the time of flight of the photon from one end to the other is to very high precision $\Delta t = L/c$. In this time the box moves a distance $\Delta x = v\Delta t = E_\gamma L/(Mc^2)$.

Now, consider how this squares up with classical physics. If you are outside the box, you don't see the goings on in the hole with the photon. As far as you are concerned, the box abruptly

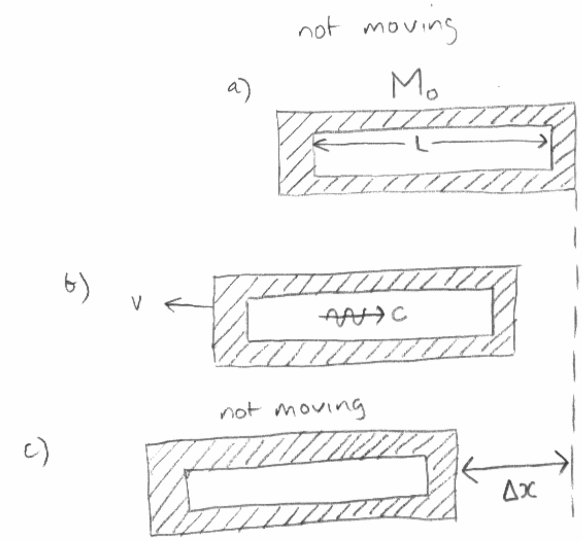

Figure 1: A photon emitted internally from one end of a cavity of length L in a box of mass M, subsequently re-absorbed at the opposite end of the box. This thought experiment was invented by Einstein to provide physical intuition for the famous result for the rest energy of a body at rest, $E_R = m_0c^2$.

starts moving at some time, then stops again a short time later. However, without an external force acting, it is not possible for the box's centre of mass to move. The only way to make everything consistent is to have the photon move some of the mass of the box from the left hand end to the right hand end when it travels. Since the box of mass M moves a distance Δx to the left, to ensure that the centre of mass does not move, a small component m of this mass must move with the photon a distance L to the right. For the centre of mass not to move overall we must have

$$\begin{aligned} mL &= M\Delta x \\ &= \frac{ME_\gamma L}{Mc^2}, \end{aligned} \tag{4}$$

and therefore

$$E_\gamma = mc^2. \tag{5}$$

Note that in deriving this result we have made a couple of approximations regarding the velocity of the box and the proportion of the mass of the box that converts to the photon. Rest assured that the result is exact even if we tighten up these assumptions. The algebra, however is harder.

This result relates the energy of a photon to some mass that was annihilated to produce it. Notice that the mass was initially at rest, before being converted into energy. From now on we will use a 0 subscript to denote a mass at rest. The result gives the equivalence between mass at rest and energy. Although in this case we have used the mass to make a photon, the energy from this mass could have also been used to create other particles, or to heat up the surroundings, for example. So we will in addition drop the γ subscript, replace it with an R subscript to remind us that this is the energy of the body at rest and simply write

$$E_R = m_0c^2 \tag{6}$$

3 Energy of a moving mass

When a massive body is moving, it will have more total energy than it does when it is at rest. We will need to know how to compute the total energy of a body from its rest mass m_0 and its velocity. We can make a guess at the answer based on non-relativistic physics. For slow moving bodies let us guess that the total energy is the sum of the rest energy m_0c^2 and the non-relativistic kinetic energy $m_0v^2/2$. The total energy is then

$$\begin{aligned} E &\simeq m_0c^2 + \tfrac{1}{2}m_0v^2 \\ &\simeq m_0c^2\left(1 + \tfrac{1}{2}\tfrac{v^2}{c^2}\right) \end{aligned} \tag{7}$$

Now let us assume that this is an approximation derived from an exact result. We also guess that the correct result should give an energy that approaches infinity as the particle's velocity approaches c. Notice that these two terms are in fact the first two terms in the binomial expansion for small v/c of

$$E = m_0c^2\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \gamma m_0c^2. \tag{8}$$

This has the desired properties, but of course we have only made a guess. The next step is to show that this guess leads to conserved quantites that correspond to those we are familiar with from pre-relativistic physics and that transformations of those quantities are consistent with those that we already found for the position and time coordinates of an event.

4 Relativistic analysis of the emission of a photon by a mass

We consider a particle of initial rest mass M_0 that emits a photon. Consider this first in the coordinate system at rest with respect to the particle, and consider the particle positioned on the positive x–axis of this coordinate system as shown in Figure 2, with the observer positioned at the origin.

On decay, the photon moves towards the observer. The energy of the particle before the decay is just its rest energy $E_i = M_0c^2$. It emits a photon of energy $E_\gamma = m_0c^2$, where m_0 is the loss in rest mass of the particle which allows the emission of the photon. The momentum of the photon after emission is

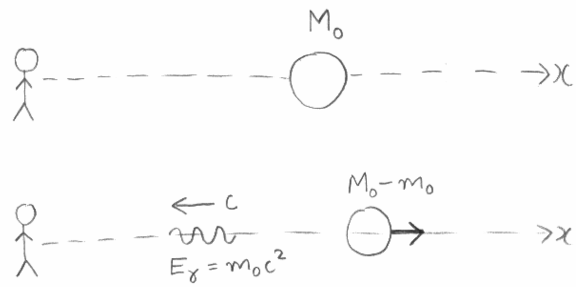

Figure 2: The emission of a photon by a massive body towards the origin. By using the known doppler shift of the photon wavelength when we transform to the coordinate system of a moving observer, we can work out the relativistic generalizations of the low-velocity approximations for the energy and momentum of a body in motion.

$$p_\gamma = -E_\gamma/c = -m_0 c. \tag{9}$$

By conservation of energy, the energy of the particle after the decay is

$$E_f = (M_0 - m_0)c^2. \tag{10}$$

By conservation of momentum, the momentum of the particle after decay is

$$p_f = +m_0 c. \tag{11}$$

Next consider the decay in the reference frame of a primed observer moving in the positive x–direction with velocity $v = \beta c$. To this observer, the particle before the decay is moving towards the primed origin with velocity $v = \beta c$. If our guess for the formula for total energy is right, its initial energy in this frame E'_i is given by

$$E'_i = \gamma M_0 c^2. \tag{12}$$

Next, consider the photon. Because in the primed frame the undecayed particle is moving towards the primed observer with a velocity $-\beta c$, the emitted photon will be blue-shifted. Using

Equation 1 with the sign of β negative, we have

$$\lambda_{\text{ob}} = \lambda_{\text{em}} \sqrt{\frac{1-\beta}{1+\beta}}, \tag{13}$$

and therefore since for a photon $E = h\nu = hc/\lambda$, we have

$$\frac{hc}{E'_\gamma} = \frac{hc}{E_\gamma} \sqrt{\frac{1-\beta}{1+\beta}}, \tag{14}$$

and hence

$$E'_\gamma = E_\gamma \sqrt{\frac{1+\beta}{1-\beta}} = m_0 c^2 \sqrt{\frac{1+\beta}{1-\beta}}. \tag{15}$$

The photon momentum in the primed coordinate system is again given by $p'_\gamma = E'_\gamma/c$, or

$$p'_\gamma = -m_0 c \sqrt{\frac{1+\beta}{1-\beta}}. \tag{16}$$

By conservation of energy the final state energy of the particle after decay in the primed frame is the difference between the initial energy in this frame, E'_i and the final photon energy E'_γ, or

$$E'_f = \gamma M_0 c^2 - m_0 c^2 \sqrt{\frac{1+\beta}{1-\beta}}. \tag{17}$$

Remember that all we have used to derive this result is a guessed formula for the total energy of a moving particle, the previously derived formula for the doppler shift of light from a moving source, and ordinary energy and momentum conservation. But is this formula consistent with the special relativity we have learned so far?

5 Lorentz transformations for energy and momentum

To answer this question, let us recall the Lorentz transformations for the ct, x, y and z.

$$\begin{aligned} ct' &= \gamma\,(ct - \beta x)\,, \\ x' &= \gamma\,(x - \beta ct)\,, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{18}$$

Let us now guess that the transformations of energy and momentum can be performed with the Lorentz transformations in a similar way. The time coordinate is replaced with energy, and the three components of the position (x, y, z) are replaced with the three components of the momentum. A factor of c is inserted so that the four quantities to be transformed have the same units, so that the four quantities corresponding to (ct, x, y, z) are $(E/c, p^x, p^y, p^z)$. The guessed transformation properties of these four components are

$$\begin{aligned} \frac{E'}{c} &= \gamma\left(\frac{E}{c} - \beta p^x\right), \\ p^{x\prime} &= \gamma\left(p^x - \beta\frac{E}{c}\right), \\ p^{y\prime} &= p^y, \\ p^{z\prime} &= p^z. \end{aligned} \tag{19}$$

Do these transformations for energy and momentum work? Let us first consider the topmost transform to calculate E' from E, p^x and c, and let us consider the final state energy of the particle in the two coordinate systems. Starting in the unprimed coordinate frame where the particle is initially at rest, the final state particle energy is given by Equation 10. The final state particle momentum is given by Equation 11. Substituting in for E and p^x in the uppermost of Equations 19, we obtain

$$\begin{aligned} \frac{E'_f}{c} &= \frac{\gamma E_f}{c} - \beta\gamma p^x \\ &= \frac{(M_0-m_0)c}{\sqrt{1-\beta^2}} - \frac{\beta m_0 c}{\sqrt{1-\beta^2}} \\ &= \gamma M_0 c - \frac{(1+\beta)m_0 c}{\sqrt{1-\beta^2}} \\ &= \gamma M_0 c - \frac{(1+\beta)m_0 c}{\sqrt{(1+\beta)(1-\beta)}} \\ E'_f &= \gamma M_0 c^2 - m_0 c^2\sqrt{\frac{1+\beta}{1-\beta}}. \end{aligned} \tag{20}$$

The last line reproduces the result of Equation 17, confirming that the total particle energy we guessed at in Equation 8 transforms, up to factors of c, the same way as the time coordinate of an event.

What about the momentum? Again, the momentum of a particle should tend to the classical value of mv in the case of low velocities, but should tend to infinity as $v \to c$. A good candidate for the momentum is therefore

$$p = \gamma m_0 v. \tag{21}$$

Again, we can check this guess against our example. In the primed frame, the initial state momentum of the particle before the photon is emitted is $p'_i = -\gamma M_0 v = -\beta\gamma M_0 c$. To conserve energy and momentum, this must equal the sum of the photon plus the particle momentum after the emission of the photon. Therefore using Equation 16 for the final photon momentum in the primed frame and requiring momentum conservation in the primed frame, we obtain

$$-\beta\gamma M_0 c = -m_0 c\sqrt{\frac{1+\beta}{1-\beta}} + p'_f. \tag{22}$$

and hence

$$p'_f = -\beta\gamma M_0 c + m_0 c\sqrt{\frac{1+\beta}{1-\beta}}. \tag{23}$$

Now let's see if this same result can be obtained from our guessed transformation for momentum, the second of Equations 19. This equation applied to the energy and momentum of the particle after emission of the photon gives

$$p'_f = -\beta\gamma E_f/c + \gamma p_f. \tag{24}$$

Using Equations 10 and 11 for E_f and p_f, respectively, we obtain

$$\begin{aligned} p'_f &= -\beta\gamma(M_0 - m_0)c + \gamma m_0 c. \\ &= -\beta\gamma M_0 c + \tfrac{m_0 c(1+\beta)}{\sqrt{(1+\beta)(1-\beta)}} \\ &= -\beta\gamma M_0 c + m_o c\sqrt{\tfrac{1+\beta}{1-\beta}}. \end{aligned} \tag{25}$$

This reproduces Equation 23. Therefore our expressions for the energy and momentum of a relativistic particle in terms of its rest mass and velocity transform just like the time and position components of a position between the values observed for different inertial observers. These energies and momenta are conserved in just the same manner as energies and momenta are conserved non-relativistically.

6 Summary

Today we have figured out relativistic expressions for the energy and momentum of a particle of rest mass m_0 moving at velocity v. They are

$$E = \gamma m_0 c^2 \tag{26}$$

and

$$p = \gamma m_0 |v|. \tag{27}$$

The transformations that give the values of energy and momentum observed by a different inertial observer are consistent with our previous knowledge of the doppler effect on the observed wavelength of light emitted by a moving source. It also turns out that the four quantities $(E/c, p^x, p^y, p^z)$ have the same coordinate transformations under change in observer velocity as the four quantities (ct, x, y, z). More about this later in the course.