Lecture 5 - Addition of velocities and relativistic space–time diagrams

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1 Review of Lecture 4

In lecture 4 we derived the Lorentz transformations relating the coordinates of the same event recorded by two different inertial (non–accelerating) observers referred to as 'primed' and 'unprimed'. Since both are inertial, their relative velocity is constant, and these transformations assumed that the primed observer is moving in the direction of the positive x axis with respect to the unprimed observer.

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ x' &= \gamma\left(x - vt\right), \\ y' &= y, \\ z' &= z. \end{aligned} \tag{1}$$

In terms of $\beta = v/c$ the Lorentz transformations can also be written

$$\begin{aligned} ct' &= \gamma\left(ct - \beta x\right), \\ x' &= \gamma\left(x - \beta ct\right), \\ y' &= y, \\ z' &= z. \end{aligned} \tag{2}$$

These transforms will be used in the next couple of lectures to explore the properties of world lines on spacetime diagrams in special relativity and to derive a couple of important kinematic results - the relativistic formula for the addition of two velocities and the relativistic formula for doppler shift.

2 Relativistic addition of velocities

Recall that in Lecture 2, we considered the example of a flashlight shone forwards out of the front of a moving train. Einstein's second special relativity postulate implies that both an observer on the train stationary with respect to the flashlight, and an observer positioned on a bridge with respect to which the train is moving at constant velocity, will measure the same speed, c, for the light from the flashlight.

In this lecture we will show that this statement follows from the Lorentz transformations. We start by considering a more general case, the case where an observer stationary with respect to the moving train train causes an object to move at a velocity u' in the direction of the forward motion of the train. The train itself is moving at a constant velocity v, and as before we define the direction of this motion to be the x axis. A second observer stationary with respect to a bridge over the railway line measures the velocity of the object moving on the train and gets a result u. In pre–relativistic physics we would expect that $u = v + u'$, but we see that this cannot be the case in special relativity, since if we replace u' by c in this formula, then $u > c$, in violation of Einstein's second special relativistic postulate.

Consider Figure 2, showing a segment of the world line of the object moving on the train. We agree that at time $t = t' = 0$, the rolling object and the origin of the primed coordinate system are in spatial coincidence with the origin of the unprimed coordinate system, so that at $t = t' = 0$, we also have $x = x' = 0$, where x represents the position of the rolling object. Some time later, this world line passes through the point (ct'_A, x'_A). Knowing these two points through which the world line passes, we may write the speed of the moving object u' as

$$u' = \frac{x'_A}{t'_A}. \tag{3}$$

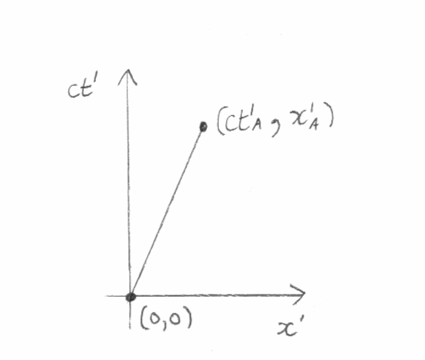

Figure 1: An object travelling at some velocity u' in the reference frame of an observer who happens to be on a train moving at velocity v with respect to a second observer. All observers are inertial.

We now use the Lorentz transformations to find the coordinates of the point A on the world line of the moving object in the unprimed frame. We get

$$\begin{aligned} t_A &= \gamma\left(t'_A + \frac{v\,x'_A}{c^2}\right), \\ x_A &= \gamma\left(x'_A + v\,t'_A\right). \end{aligned} \tag{4}$$

Next we write $x'_A = u't'_A$ in these equations and factor out the common $\gamma\,t'_A$,

$$\begin{aligned} t_A &= \gamma\,t'_A\left(1 + \frac{u'\,v}{c^2}\right), \\ x_A &= \gamma\,t'_A\left(u' + v\right). \end{aligned} \tag{5}$$

Dividing the lower of these two equations by the upper yields a formula for u, the velocity of the moving object on the train as measured by an observer on the bridge.

$$u = \frac{x_A}{t_A} = \frac{\gamma\,t'_A\left(u' + v\right)}{\gamma\,t'_A\left(1 + \frac{v\,u'}{c^2}\right)}. \tag{6}$$

Cancelling the common factor of $\gamma\,t'_A$ yields our final result for the velocity of the moving object as measured on the bridge.

$$u = \frac{u' + v}{1 + \frac{u' v}{c^2}}. \tag{7}$$

Now let us see when the moving object on the train is a light pulse. This means that $u' = c$. Substituting in to Equation 7, we obtain

$$\begin{aligned} u &= \frac{c+v}{1+\frac{vc}{c^2}} \\ &= \frac{c+v}{1+\frac{v}{c}} \\ &= \frac{c+v}{\left(\frac{c+v}{c}\right)} \\ &= c, \end{aligned} \tag{8}$$

so that to the observer on the bridge, the velocity of photons emitted by a source on the train is c, just as it is for an observer stationary with respect to the light source.

So is the light emitted by the flashlight on the train the same in all respects to both the train and the bridge observer? No, in fact the light measured by the two observers will in fact have different frequencies, wavelengths, and energies per photon in their two frames of reference, due to the Doppler effect. You have met the Doppler effect before in the context of sound waves emitted by moving sources. The simplest example is the sound made by sirens on moving ambulances which seems to have a higher pitch as the ambulance approaches and a lower pitch as the ambulance moves away. We will learn about the Doppler effect in special relativity very soon. For now, note only that unlike the doppler shift of sound waves from relatively slow ambulances, relativistic Doppler shift occurs even when the oscillator has motion only transverse to the line of sight between the observer and the moving source. But before considering the Doppler effect, let us return to spacetime diagrams and world lines now that we have the Lorentz transformations at our disposal.

3 Review of pre-relativity spacetime diagrams

Recall from the first lecture that we can plot events in spacetime on a diagram of time versus position. Considering again the

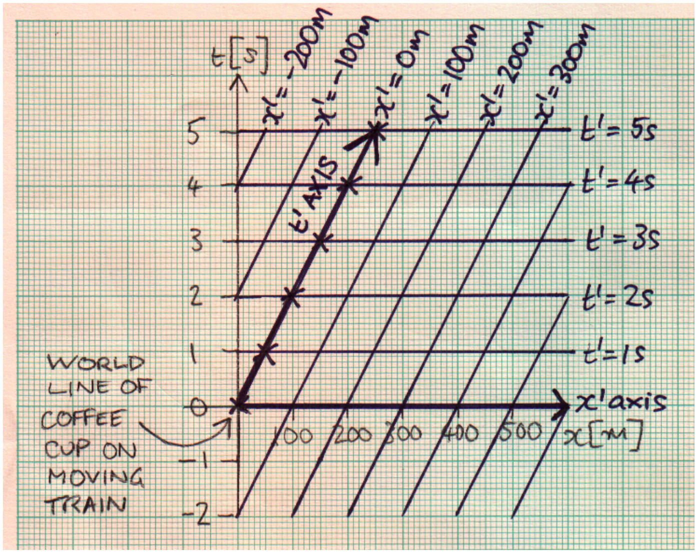

Figure 2: Overlay of coordinate systems of two observers predicted by the pre-relativistic Galilean transformations. Notice that lines of equal time are coincident between the two coordinate systems since clocks run at a rate independent of their speed in pre-relativity physics.

coffee cup on the train, two observers called the primed (on the train) and unprimed (on the bridge) observers both view the coffee cup. In the inertial coordinate system of the bridge observer, the sequence of events making up the world line of the coffee cup forms a diagonal line with the slope of the line equal to $1/v$, where v is the velocity of the train. In the world line of the observer on the train, the coffee cup's world line follows the line $x' = 0$, since the coffee cup stays fixed at the origin of the primed coordinate system

This can be made consistent by overlaying lines of constant t' and x' in a spacetime diagram of the world line of the coffee cup in the coordinate system of the unprimed observer. This is illustrated in Figure ** . The line $x' = 0$ follows a diagonal line of gradient $1/v$, where v is the velocity of the train. The world line of the coffee cup follows this line; this is not surprising since in the primed coordinate system, the coffee cup is at the origin at all times. The important thing to note here is that the method for finding the t' and x' axes was to use the Galilean transformations and substitute $x' = 0$ and $t' = 0$, respectively. When we repeat this trick with the Lorentz transformations, we will find out how to overlay the coordinate axes of the primed observer on top of spacetime diagrams of the unprimed observer in special relativity.

4 The Lorentz version of the axes of Spacetime diagrams

First, as mentioned in lecture 4, note that if x and t are in SI units, ie, 1 on the x–axis means 1 m, and 1 on the time axis means 1 s, then the world line of a light pulse is phenomenally close to horizontal, since in 1 s a light pulse travels 3×10^8 m. If we are considering relativistic effects, having light pulse world lines so close to flat makes it hard to develop an intuition for the consequences of the time it takes for light to travel from place to place. And another thing - the dimensions (units) of the time axis and the x–axis are different.

To fix both of these problems, we make the vertical axis equal to the time coordinate times c, the speed of light. So relativistic spacetime diagrams are plots of ct versus x. What does the world line of a light pulse look like in these coordinates? Consider a light pulse that starts out at time zero at the origin and subsequently propagates in the direction of increasing x. In a time t it travels a distance ct, so that for any point on its trajectory after time zero, $x = ct$. On a graph of ct vs x, therefore, the world line of this light pulse is a straight line through the origin with gradient 1; compare $ct = x$ with the general equation of a straight line $Y = MX + C$, and you get $Y \rightarrow ct$, $X \rightarrow x$, $C \rightarrow 0$, $M \rightarrow 1$. Therefore the gradient of the line is 1, and its ct (or Y) axis intercept is zero. The world line of this photon is shown in Figure 3. Also shown on this figure is the world line of a second light pulse that starts out at the origin but travels in the opposite direction, that of decreasing x.

5 The future light cone

Now why have I shaded in the triangle between the world lines of the two photons and labelled this sector 'causual future of an event at the origin'? Remember that we decided nothing can travel faster than the speed of light? So the shaded area

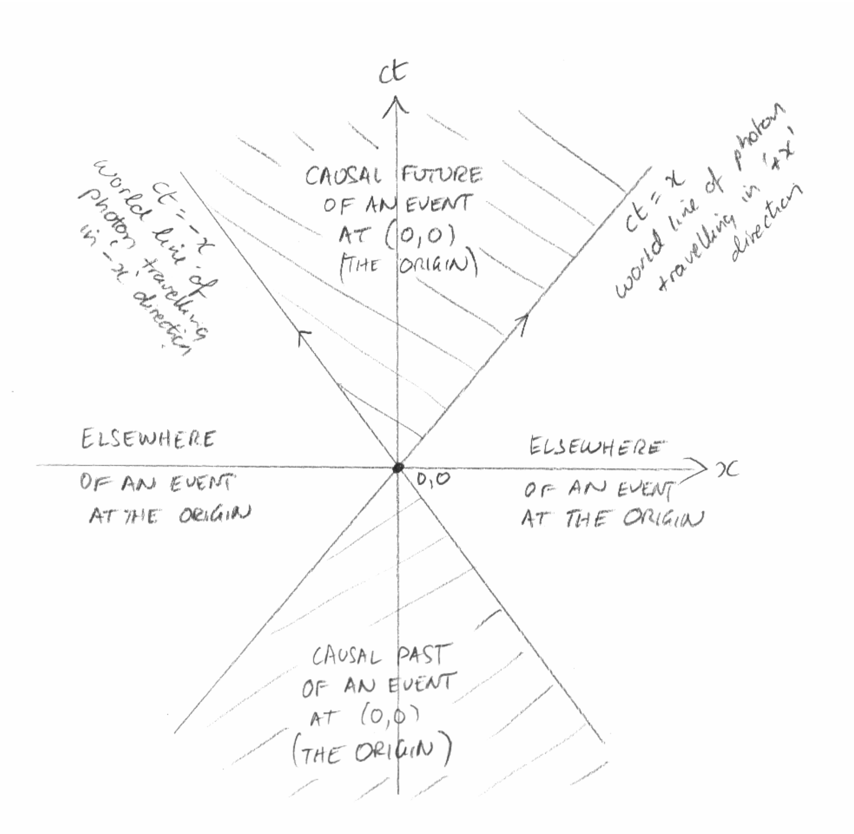

Figure 3: The world lines of photons passing through the origin in the reference frame of some observer. The causal past of this observer is all the events that can have a causal influence on the origin. The causal future is all the events on which the origin event can exert a causal influence. Points outside the shaded areas can have no causal influence on the point $(ct = 0, x = 0)$; this set of points is called 'space-like separated' from the origin, or sometimes 'elsewhere'.

represents the set of points in spacetime that a particle (or some information, or anything else) travelling at a speed allowed by relativity can reach. Any world line connecting an event at the origin with a point outside this shaded region is unphysical - any world line connecting these two points has to have gradient less than 1 at some point along it, which means that the world line represents something moving faster than the speed of light, something that is ruled out in special relativity. Any point inside this shaded region can be reached from the origin by means of a world line that never has slope less than 1, so that the world line represents something propagating at a speed always less than c.

The set of shaded points is often called the forward light cone. The reference to a cone has to do with the fact that where you allow two spatial dimensions, the set of world lines representing light pulses starting at the origin and moving along straight line trajectories in the x–y plane forms an upside down cone, with every event inside the cone reachable by a particle travelling at less than the speed of light and starting at the origin, and everything outside the cone unreachable from the origin in special relativity. We say therefore that every event in the future light cone of the origin is causally connected to the origin. If a signal can reach that event from the origin, an event at the origin is capable of exerting a causal influence on the later event.

What about the future light cone of events not at the origin? What is the set of points in spacetime causally connected to an arbitrary event? On a spacetime diagram it is easy to find this set of points. Simply draw the two straight lines at 45° above the horizontal to the left and to the right of the event in question? Any event between these two lines is in the causal future of the point; any event not between these two lines is not in the causal future of the event. There is nothing somebody at that point can do to influence what happens inside his or her future light cone.

6 The past light cone

Similarly, one may draw lines at 45° approaching an event from the past. The region bounded by these two lines is called the causal past or the past light cone of the event. The set of points in this region is the set of all the events which may exert a causal influence on the event in question. Any event outside this region can only be connected with this point by a world line that moves faster than the speed of light at least some of the time, and is therefore unphysical in the context of special relativity.

7 Elsewhere

You will notice that the past light cone and the future light cone of an event do not cover all of the ct–x plane. There is a large region of spacetime that is neither in the past nor the future light cone of any given event. This region is sometimes called 'elsewhere', though of course it is only 'elsewhere' for a particular event. Points in this region represent events that may occur at an earlier time or a later time than the origin event, but are too far away from the origin to either causally influence the event or be causally influenced by it.

8 Space–like and time–like separations of events

Two events are said to be time–like if one event is in either the future or the past light cone of the other. If this is not true, then each event is in the elsewhere of the other. In this case, the two events are also said to be space–like separated.

9 The sequence of time–like separated events

If two events are time–like separated, then a signal may propagate from one event to another. From a philosophical standpoint, this is connected to the concept of causality. All observers must agree that the two events occurred in the same order. Suppose event A is my releasing a stone from my hand, and event B is the stone breaking the window of a greenhouse. All observers must agree that event A precedes event B in time, since it is nonsense to think about the shattering glass of the greenhouse causing the stone to be released from my hand. And we can see that these types of event pairs have to be time–like separated because the world line of the stone travelling from my hand to the greenhouse window corresponds to a stone moving at less than the speed of light.

10 No agreed sequence for space–like separated events

If two events are space–like separated, then no signal travelling at a speed allowed by special relativity may connect the two events, and therefore one event cannot cause the other. In principle, and it turns out in practice, there is no reason why two different inertial observers have to agree that the two events occurred in the same order. So suppose myself and my friend both agree to release stones at the same time in our coordinate system at rest with respect to the greenhouse, but at different points, Suppose further that both stones arrive at the greenhouse at the same time as measured in the inertial coordinate system we share with the greenhouse. Now consider what other observers moving at different velocities might conclude. It will turn out that as long as the events of the stone releases from our hands are space–like separated, some observers see me release my stone before my friend releases his, and others see my friends stone start out first. To see this we will use space–time diagrams.

11 Overlayed coordinate axes in special relativity

Let us see now how to overlay coordinate axes for a primed coordinate system in a spacetime diagram in unprimed coordinates. Suppose that the primed frame is again moving in the direction of increasing x with respect to the unprimed frame. Then Equations 2 are the Lorentz transforms between the two frames. As in the Galilean case we find the equation for the ct' axis by writing $x' = 0$ into the Lorentz transform for x' to obtain the line $x = vt$, or $ct = cx/v$, or $ct = x/\beta$, where $\beta = v/c$ as

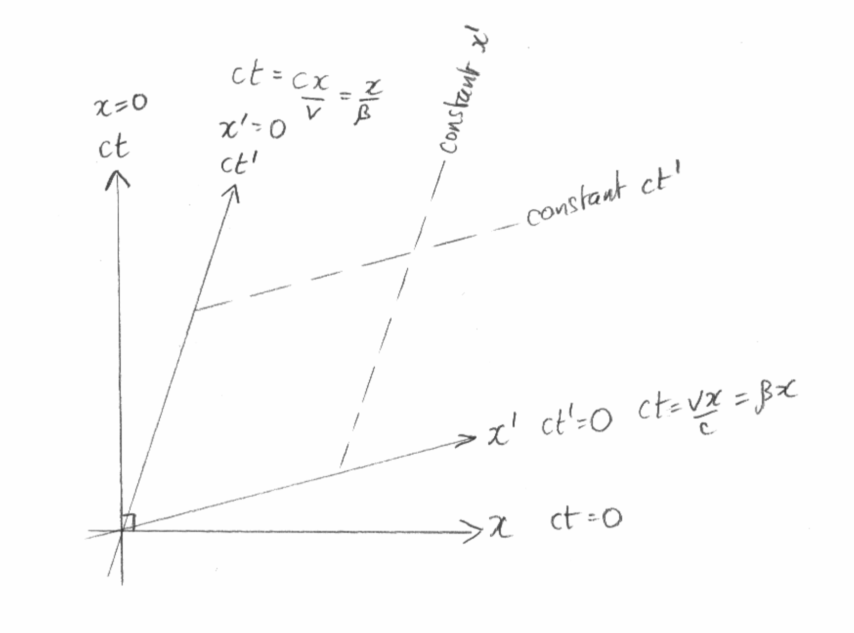

Figure 4: The coordinate systems of two inertial observers overlaid. Lines of constant t' are not parallel to lines of constant t. As a consequence, different inertial observers disagree about the time ordering of events that are time-like separated. This did not occur in pre-relativity physics.

before. This is a straight line through the origin with gradient $1/\beta$. Similarly, the x' axis is the line $ct' = 0$, which means that $ct = \beta\, x$, the equation of a straight line with gradient β through the origin. So the ct' and x' axes have gradients which are reciprocals of each other when drawn in the ct–x plane. Once again the primed coordinates are skewed with respect to the unprimed ones. This is illustrated in Figure 4.

Notice also in this figure that lines of constant ct' are at different angles with respect to the horizontal lines of constant ct. This illustrates geometrically the point that in special relativity two inertial observers may disagree with the order of two events that are space-like separated. Consider; if two events are space-like separated, then they can be joined together by a straight line of slope less than 45 degrees. In this case, I can always find an observer for whom a line of constant ct' can have a slope steeper than this line. For this observer, the two events will occur in the temporal order BA. However, for another observer, whose lines of constant ct'' are of a less steep slope than the straight line joining A and B, the two events occur in the order AB. This ceases to be true of events that are time-like separated, since such events are linked by a straight line of gradient greater than one. The maximum slope of a line of constant ct' is 1, for an observer moving infinitely close to the speed of light, and even this observer does not see the order of the two events reverse.