Lecture 4 - Lorentz contraction and the Lorentz transformations

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1 The inadequacy of the Galilean transformations

In Lecture 1 we learned that two inertial (non-accelerating) observers, one of which is moving at constant velocity v in the direction of increasing x, measuring the time and position of the same event obtain answers that, in pre-relativity physics were thought to be related by the Galilean transformations,

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z, \end{aligned} \tag{1}$$

where the coordinates of the event to the unprimed observer are (t, x, y, z), and the coordinates of the same event to the primed observer are (t', x', y', z'), and the primed observer has a velocity of $+v$ with respect to the unprimed observer aligned parallel to the x–axis.

In lecture 2, however, we learned that clocks moving with respect to an observer run slow by a factor of $\gamma = 1/\sqrt{1 - v^2/c^2}$ with respect to clocks stationary with respect to an observer. This means that the Galilean transformations must only be approximate, and must be special cases of more general transformations. It is easy to show this. Suppose you have a clock, which is moving at velocity v along the +ve x–axis with respect to the unprimed observer, but is stationary with respect to a primed observer moving with the clock. Let events 1 and 2 be two successive ticks of the clock. In the unprimed frame, the times of these two events are t_1 and t_2, and in the primed frame they are t'_1 and t'_2 respectively. The time between clock ticks in the unprimed frame is $\Delta t = t_2 - t_1$, and in the primed frame is $\Delta t' = t'_2 - t'_1$. But since $t' = t$, we get $\Delta t' = \Delta t$, and the Galilean transformations predict that a moving clock runs at the same speed as a stationary one, in contradiction with Einstein's theory.

2 Deducing the Lorentz transformation for x

Now let us use our knowledge of the time dilation formula to guess a form for the transformation between x and x', and then having guessed this form, shore up our guess by thinking more about the consequences of time dilation. Time dilation scales time intervals by a factor of γ, so perhaps it scales spatial intervals as well. Let us start with the Galilean transformation $x' = x - vt$, and guess the form of a more general transformation that allows time and spatial coordinates to be scaled by some initially unknown factors B and A, respectively. Where B and A equal 1, we get back to the Galilean transformations.

$$x' = Ax - Bvt \tag{2}$$

Suppose that two observers agree to locate the origins of their coordinate systems at the origin at time $t = t' = 0$, and that the primed observer and her coordinate system are moving at velocity $+v$ with respect to the unprimed coordinate system. Perhaps the primed observer is moving with the muons that are decaying in our atmospheric experiment. At all subsequent times, the muons remain at the origin in the primed coordinate system. For this to be the case, we must have $A = B$ in Equation 2, so the transformation equation becomes

$$x' = A(x - vt). \tag{3}$$

Now, recall from the end of the movie, the question of what an observer moving with the decaying muons sees. To this observer, the muons are at rest, and so they decay at their normal rate, and the muon clock must run at the same speed as his watch. Yet when this observer reaches the bottom of the mountain, the fraction of the muons left undecayed must agree with the fraction of undecayed muons measured by the observer stationary with respect to the Earth. The only possible way to reconcile the two observers is if the distance travelled by the muons as observed in coordinates at rest with respect to their motion must appear smaller by a factor of gamma.

Let's cast this statement in mathematical terms. Let us say that the unprimed coordinate system is that at rest with respect to the decaying muons, and the primed coordinate system at rest with respect to the Earth. The distance moved by the muons as measured in their rest frame is L, the height of the mountain. The distance moved by the muons as measured in the rest frame of the Earth is L'. This L' is the ordinary height of the mountain as it appears on a map. We have argued that we must have

$$L = \frac{L'}{\gamma} = L'\sqrt{1 - \frac{v^2}{c^2}}. \tag{4}$$

This result is the Lorentz contraction of length of an object in the direction of motion of an observer with respect to whom the object is moving. In this case, the direction of motion of the muons is downwards, so that the height of the mountain is reduced in the reference frame of the muons.

We have deduced the necessity of Lorentz contraction from the phenomenon of time dilation. An observer at rest with respect to Earth sees that more muons survive to ground level than is predicted if one assumes that the muon lifetime is the same as its lifetime at rest. This observer infers that the mean life of the muons is extended by a factor of γ. The assembly of muons is a moving clock running slowly. An observer at rest with respect to the muons sees the same number of muons surviving to ground level, yet to this observer the muons are at rest and have an unaltered mean life. The only way this can be true is if, to the observer moving with the muons, the vertical distance through which the muons must move to get to ground level, the height of the mountain, is contracted by the same factor, γ.

3 The Lorentz Transformations

Now let us see if we can use Equation 4 to deduce the value of the constant A in the generalized coordinate transform for x in Equation 3. Again, we regard the primed observer as stationary with respect to the Earth, and the unprimed observer as stationary with respect to the muons. We have $x' = A(x - vt)$. Now, the length of an object is the difference in the coordinates of its two ends. Let us call the coordinates of the top and bottom of the mountain x_1 and x_2 when measured by the unprimed observer (the observer at rest with respect to the Earth), and x'_1 and x'_2 when measured by the primed observer (at rest with respect to the muons). To determine the height of the mountain, the unprimed observer (in the rest frame of the muons) measures x_1 and x_2 at the same time t_m. The mountain is moving towards the muons at velocity v directed in the $-x$ direction. Inserting these quantities into Equation 3 we obtain two equations for the positions of the base and top of the mountain as measured in the rest frame of the mountain.

$$\begin{aligned} x'_2 &= A(x_2 - (-v)t_m) \\ x'_1 &= A(x_1 - (-v)t_m) \end{aligned} \tag{5}$$

Subtracting the second of these equations from the first we obtain

$$(x'_2 - x'_1) = A(x_2 - x_1) \tag{6}$$

But $x'_2 - x'_1 = L'$ and $x_2 - x_1 = L$, and hence we have $L' = AL$, or $L = L'/A$. Comparing this with Equation 4 we deduce that $A = \gamma$. The equation for the transformation of the spatial coordinates of an event between two coordinate systems is

$$x' = \gamma(x - vt), \tag{7}$$

where the primed observer is moving at velocity v directed along the positive x–axis with respect to the unprimed observer. This is the first of the Lorentz contractions. What about the time coordinate? Notice that if the primed frame is moving at a velocity $+v$ with respect to the unprimed frame, then the unprimed frame is moving at a velocity $-v$ with respect to the primed frame. Therefore we can write the expression for x in terms of x' and t' based on Equation 7.

$$x = \gamma(x' + vt'). \tag{8}$$

Substituting Equation 7 into equation 8 we obtain

$$x = \gamma \left(\gamma(x - vt) + vt'\right) \tag{9}$$

We can rearrange Equation 9 to express t' in terms of x and t:

$$\gamma v t' = (1 - \gamma^2)x + \gamma^2 vt, \tag{10}$$

or

$$t' = \gamma t + \frac{(1 - \gamma^2)}{\gamma v} x. \tag{11}$$

This can be further simplified by writing

$$\begin{aligned} 1 - \gamma^2 &= 1 - \tfrac{1}{1-\frac{v^2}{c^2}} \\ &= \tfrac{\left(1-\frac{v^2}{c^2}\right)-1}{1-\frac{v^2}{c^2}} \\ &= \tfrac{-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} \\ &= \tfrac{-v^2\gamma^2}{c^2}. \end{aligned} \tag{12}$$

Therefore

$$\frac{1 - \gamma^2}{v\gamma} = \frac{-\gamma v}{c^2}. \tag{13}$$

Substituting Equation 13 into equation 11 we obtain

$$t' = \gamma \left(t - \frac{vx}{c^2}\right). \tag{14}$$

Equation 14 gives the Lorentz transformation for the time coordinate of an event. Let us finally group together this equation, the earlier result (Equation 7) for the transformation of the x–coordinate, and the remaining transformations $y' = y$ and $z' = z$ since the relative motion along the x–axis has no effect on coordinates in perpendicular directions, to obtain the full set of Lorentz transformations of coordinates.

$$\begin{aligned} t' &= \gamma\left(t - \tfrac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z, \end{aligned} \tag{15}$$

where the primed coordinate system is moving with a velocity v directed in the positive direction along the x–axis with respect to the unprimed coordinate system, and where $\gamma = 1/\sqrt{(1 - v^2/c^2)}$. These equations are collectively called the Lorentz transformations. They will be the basis of much of the remainder of the course.

4 What is meant by coordinates?

Before figuring out some of the consequences of the Lorentz transformations, let us have a quick think about what is meant by coordinates. This is less obvious in the case of these new transformations than it was in the case of the older Galilean ones, because in the new transformations the time on clocks moving at different velocities passes at different rates. So we have to be more careful about what we mean by time than we were before.

So here is what is meant by the time of an event to some observer, say the unprimed observer. The observer possesses a cartesian grid of points. This grid could actually be constructed, say out of metre rules with their ends joined in a simple cubic lattice. The spatial coordinates of any event are found using the rulers by reading off each of the three components. This much is pretty clear.

Now for the time component. Here we have to use our imagination. We put a clock at each of the vertices of our grid, and arrange for all these clocks, which are all at rest in the same inertial frame, to be synchronized. See Figure 1 for an illustration of this construction. There is no problem with synchronizing all the clocks together, as you might at first worry. Suppose you send out a light pulse in all directions from a single point in the coordinate system. For sure this pulse will take longer to reach the clocks that are further away than the nearer ones. But you compensate for this effect by setting the initial time on the further away clocks to be later than on the nearer clocks by the right amount. The light pulse propogates outwards, starting all the clocks as it goes, and after a while all the clocks are running at the same rate (because they are all in the same inertial frame), and there is no time offset between them. So all the clocks are now synchronized. Fabulous

Now, suppose we want to know the time of some event. It's the time on the nearest clock to the position of the event when the event happens. If you need a more accurate determination, build a finer grid with more clocks closer together. The reason for this construct is to emphasize that the effect of moving clocks running slow, or more generally of the difference between the time coordinates of the same event in different inertial frames has nothing to do with the finite propagation velocity of the speed of light to some remote observer. It is not the case that the time of an event is the time at which the light from the event reaches an observer at the origin of the coordinate system. No, there is no special origin and time is a quantity that can be measured just as well locally at any point in the coordinate system. A zero of time is chosen arbitrarily just as a zero position (the origin) is chosen, but these zeros have no physical significance except for an additive constant in the determination of the coordinates of events.

5 An example - Analysis of charge teleportation with Lorentz transformations

Back to charge teleportation, and let us see whether this holds up in the world or Lorentz transformations. Suppose, to an

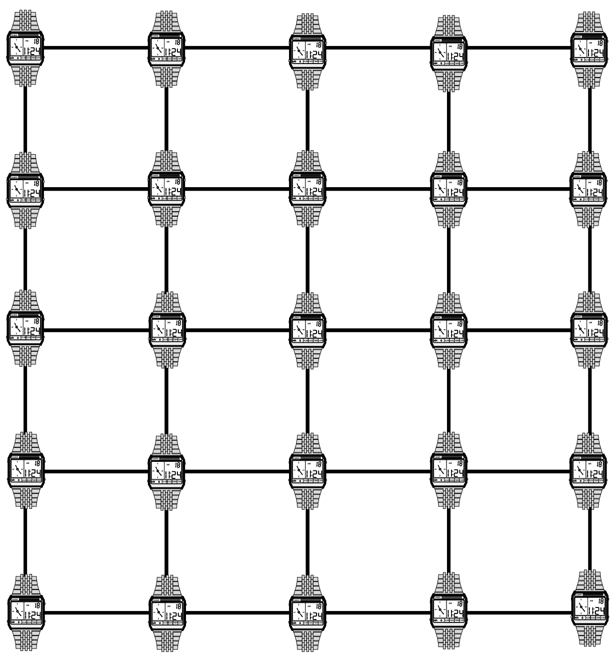

Figure 1: A coordinate system for special relativity. The framework of points represents a cartesian coordinate system, here shown in two dimensions. At each vertex is a clock. The clocks are all synchronised.

observer stationary with respect to the Earth, an electron were able to disappear at time $t = 0$ and appear at the same time in a nearby galaxy, say at a distance of 20 Mpc which is 6.16×10^{23} [m]. The coordinates of the reappearance in this coordinate system would be $t = 0$ [s] and $x = 6.16 \times 10^{23}$ [m]. Now suppose an observer is moving at a velocity of 1/2 of the speed of light. Let us calculate γ for this observer. It is $\gamma = 1/\sqrt{(1 - 1/4)} = 2/\sqrt{3} = 1.15$. The timescales in the two coordinate systems have their origins set to coincide at the disappearance of the electron on earth, so for the disappearance $t_0 = t'_0 = 0$ [s]. Let us figure out the t' coordinate of the event of the electron reappearing to this observer.

$$\begin{aligned} t'\,[\mathrm{s}] &= 1.15\left(0\,[\mathrm{s}] - \tfrac{0.5\times 3\times 10^8\,[\mathrm{m/s}]\times 6.16\times 10^{23}\,[\mathrm{m}]}{(3\times 10^8\,[\mathrm{m/s}])^2}\right) \\ &= -1.18 \times 10^{15}\,[\mathrm{s}] \\ &= -37 \times 10^6\,[\mathrm{years}]. \end{aligned} \tag{16}$$

To this observer, then, the electron re-appears in the Virgo cluster (about the right distance) 37 million years before it vanished on Earth! To this observer, then, charge is not conserved, since for 37 million years there are two electrons in the Universe where before and after this time interval there were only one. And you can see that had the primed observer been moving in the direction of decreasing x, The charge would have vanished altogether for this time period, and then reappeared. This is absurd, and to get away from these kinds of problems, we end up concluding that nothing can travel faster than the speed of light. This prevents any inertial observer from seeing events occur outside of their causal sequence; if one event causes another, then to all observers the event that does the causing must happen before the event it causes. However, different inertial observers can disagree about the sequence of events that are not causally related. More about this later in the course.