Lecture 2 - First principles of special relativity and time dilation

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1 Einstein's postulates

Last time we learned about events and world lines on spacetime diagrams, about the non-relativistic transformations between the coordinates (t, x) and (t', x') of the same event measured by two different inertial observers, and about relating spacetime diagrams of the same world line drawn by different inertial observers. Finally, we noted that there is a problem in classical, non-relativistic physics. That is, in principle teleportation of objects is allowed, which seems to be nonsense.

Teleportation can be thought of as the propagation of a body from one position to another at infinite speed. Einstein's theory approaches the problem of infinite speeds in the second of his two postulates. Rather surprisingly, he makes a particular speed, the speed of light, a special quantity. Here are his two postulates stated formally:

1. All inertial frames are equivalent with respect to the laws of physics.
2. The speed of light in a vacuum is the same to all observers: $c = 3.0 \times 10^8\,\mathrm{m\,s^{-1}}$.

Postulate 1 is a restatement of the fact that there is no absolute rest frame. It does not mean that two inertial observers will always get the same result when they make a measurement of the same system. Instead, it means that the results of experiments performed entirely in an inertial lab will be independent of the velocity of that lab. Rather subtly, this postulate incorporates the requirement that two inertial observers agree as to their relative velocities, although if observer 1 measures the velocity of observer 2, and vice versa, their two results will differ by a sign. Postulate 2 is the revolutionary one. Consider its consequences. You are on a train and you shine a light out of the front window. You measure the velocity of the light beam emerging from the torch by some means. You get $3.0 \times 10^8\,\mathrm{m\,s^{-1}}$. Your friend on a bridge measures the velocity of the photons with respect to the bridge. He also gets $3.0 \times 10^8\,\mathrm{m\,s^{-1}}$ and not the sum of this value and the velocity of the train. So for a start, the pre-relativistic idea of addition of velocities is in need of modification to be consistent with postulate 2. Einsteins postulates have other consequences, including the elimination of the charge teleportation problem alluded to in lecture 1. However, let us start with time dilation and return to teleportation later on.

2 A clock that uses light

Let us go into the business of making reliable clocks, something needed for all mechanics experiments. We have in postulate 2 the idea that something (light) always travels in vacuum at the same speed. Let us try and build a clock that exploits this fact, and see where it leads us. Figure 1 is a diagram of the clock.

The clock consists of a source of light pulses with a coincident light sensor, and a reflecting mirror a distance L away. A short burst of light is fired towards the mirror, bounces, and returns to be registered by the sensor after a travel time of

$$\Delta\tau = 2L/c. \tag{1}$$

The time of flight of the light pulse $\Delta\tau$ is the time between clock ticks.

Why build a clock like this? Firstly because it uses light whose speed (we are told) is the same to all obsevers. Secondly, because its mechanism is entirely one dimensional, so that if we cause the clock to be moving with respect to another inertial observer, and the direction of motion is perpendicular to the light path in the instrument, we are pretty sure that the instrument will

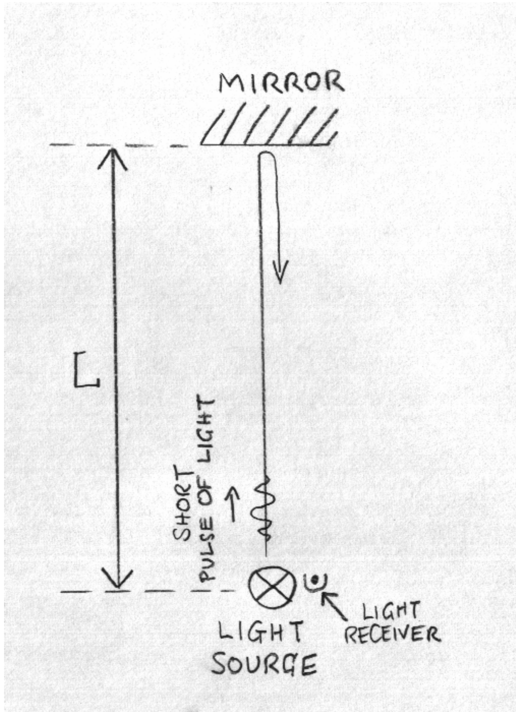

Figure 1: A light clock stationary with respect to our observer. The time interval between clock tics is taken to be the time of flight of a light pulse between the source and the photodetector, bouncing off the mirror on the way.

not be distorted spatially from the point of view of the observer with the motion with respect to the clock. Let us then suppose that the clock is moving to the right with respect to a second observer, at velocity v. The whole clock (light source, mirror, and light sensor, moves in unison. This is illustrated in Figure 2.

3 Derivation of the time dilation formula

Let us say that to this observer it takes a time $\Delta t'$ for the light in the moving clock to make the round trip. We will not assume that $\Delta t' = \Delta t$ as we would in pre-relativity physics. We infer that it took $\Delta t'/2$ to reach the end mirror, by symmetry. In this time the clock would have moved $v\Delta t'/2$, where v is the velocity of the clock with respect to the observer. The vertical

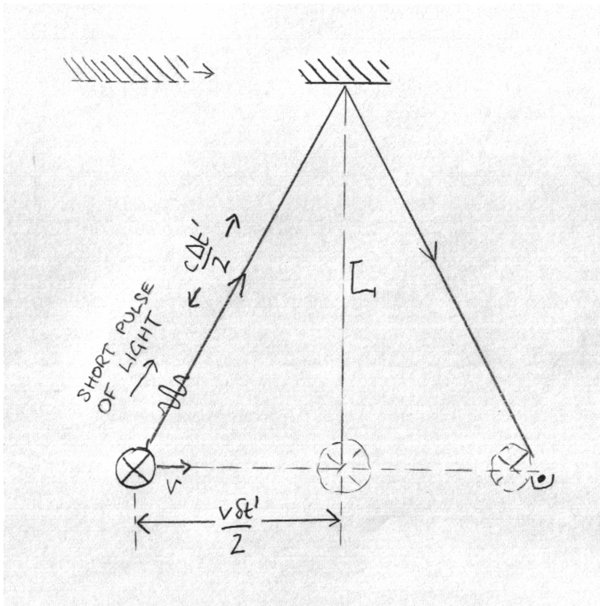

Figure 2: Our light clock as seen when it is moving at velocity v to the right with respect to our observer.

displacement of the light between the source and the mirror is unchanged - it is still L. Therefore by Pythagoras' theorem, the length of each of the two diagonals is $\sqrt{L^2 + v^2\Delta t'^2/4}$. Therefore the total distance travelled in time $\delta t'$ is $2\sqrt{L^2 + v^2\Delta t'^2/4}$. But the distance travelled divided by the time taken is the speed of the light pulse, which by postulate 2 must be c. Therefore we get

$$c = \frac{2\sqrt{L^2 + \frac{v^2\Delta t'^2}{4}}}{\Delta t'}. \tag{2}$$

Therefore

$$\Delta t' = 2\sqrt{\frac{L^2}{c^2} + \frac{v^2\Delta t'^2}{4c^2}}. \tag{3}$$

Substituting in for L/c from Equation 1 we obtain

$$\Delta t' = 2\sqrt{\frac{\Delta\tau^2}{4} + \frac{v^2\Delta t'^2}{4c^2}}, \tag{4}$$

or,

$$\Delta t' = \sqrt{\Delta\tau^2 + \frac{v^2\Delta t'^2}{c^2}}. \tag{5}$$

Squaring both sides we obtain

$$\Delta t'^2 = \Delta\tau^2 + \frac{v^2}{c^2}\Delta t'^2. \tag{6}$$

Rearranging and taking the square root, we obtain our final result

$$\Delta t' = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{7}$$

It is common to define γ as

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{8}$$

because this factor comes up so often in relativity. In this notation, Equation 7 is written

$$\Delta t' = \gamma\Delta\tau. \tag{9}$$

This result is called the time dilation formula. What are it's components? $\Delta\tau$ is the time between ticks of a clock (albeit an odd one) measured by an inertial observer in whose coordinate system the clock is at rest. $\Delta t'$ is the time between clock ticks measured by a second inertial observer, with respect to whom the clock is moving with a constant velocity v directed along the x axis, in this case in the +ve x direction. This formula has various interesting properties. First, note that if $v > c$, the square root has no real solution. Let us therefore ignore this case for now, and consider $v < c$. For all $v < c$, the square root is less than one, therefore $\Delta t'$ is greater than Δt. Third, the size of the effect does not depend on the sign of v, only on its magnitude. It does not matter whether the clock is moving to the right or to the left with respect to the observer, it is only the magnitude of v that determines the amount of slowing. Since we chose the x axis arbitrarily, the clock can actually be moving in any direction and you will get the same slowing of the pulse rate, dependent only on the magnitude of v. And, as v approaches c the clock tends to stop.

4 The behaviour of the gamma factor

The γ factor determines how much slower the clock moves when it is moving with respect to the observer than it does when it is at rest. Therefore we should look at how γ depends on v. In Figure 3 we have plotted γ against v/c, which is commonly written β. When β is 1, $v = c$. When $\beta = 0$, the clock is at rest.

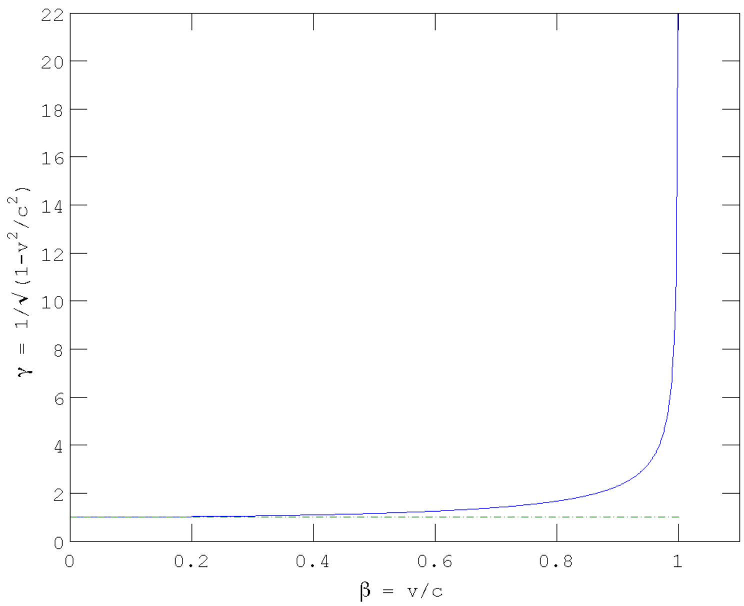

Figure 3: A plot of the relativistic γ factor as a function of v/c, or β.

Notice that γ is very close to one until the clock speed passes about 1/3 of the speed of light, but that as v approaches c, γ becomes infinite. Notice also that $\gamma > 1$ for all v.

5 Assemblies of radioactive particles as clocks

To a pre-relativity classical physicist, or to a sceptic (and there are still some!), time dilation seems crazy. It defies common sense. However, in another famous quote, this time from Einstein:

Common sense is the collection of prejudices acquired by age eighteen

Rather as in quantum mechanics, there is a danger here of letting our everyday experience cloud our judgement of physics outside this realm. Let us instead if experiments support the prediction of time dilation.

The problem with testing relativity in the lab is that its effects such as time dilation only come into play when the objects in question, clocks in this case, are moving at speeds close to the speed of light. To test the prediction of time dilation we need a clock that is moving much faster than any speed which we are capable of accelerating an alarm clock to, for example. Fortunately, we have at our disposal a wonderful array of natural clocks - subatomic particles that are light enough so that they commonly move at high speeds, and have an inbuilt process associated with them that has a natural measurable timescale. The classic example is particles unstable to radioactive decay, or just radioactive particles. You will have met these at A level, and there are a great variety of them. A given species of radioactive particle has a characteristic lifetime, sometimes expressed in terms of the mean lifetime τ of the particle. If you have an assembly of N_0 radioactive particles having a mean life τ, then after a time t the number $N(t)$ of particles left undecayed is given by the radioactive decay law:

$$N(t) = N_0 e^{\frac{-t}{\tau}}. \qquad (10)$$

For example, muons (or mu-mesons, or just μ^+, μ^-) are unstable particles a bit like electrons, but heavier, that have a mean life of 2.2 microseconds. If you start out with, say 1000 of these particles, and wait for 10 microseconds, Figure 4 shows the number $N(t)$ of the particles left undecayed as a function of time over this interval, from Equation 10.

The decay of an individual particle cannot be used as a clock, because the exact moment of decay is uncertain. However, if we have an assembly of muons that is sufficiently large, we can make a clock out of the whole assembly. Suppose initially we have 1000 muons in a box. Some (undetermined) time later, we count how many are left. This tells us how much time is elapsed. For example, if we open the box and discover that 400 muons

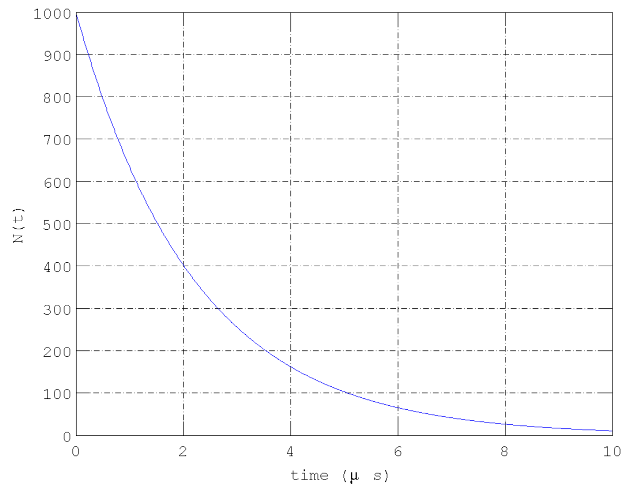

Figure 4: The number of undecayed muons remaining from an initial population of 1000 as a function of time over 10 μs.

remain undecayed, Figure 4 tells us that 2 μs have elapsed since we started.

Now, suppose we want to use this clock to test time dilation. We would arrange for these muons to be moving at a known velocity with respect to us, we would then count how many muons we have at some initial instant, wait a prescribed amount of time as measured on our own, stationary clock, and see how many muons are left. The number of muons left tells us how much time has elapsed on the moving clock made out of the collection of high velocity muons. Our own trusted clock tells us how much time has elapsed to a stationary clock. The ratio of the moving clock time to the stationary clock time is equal to γ. If we know the velocity of the muons, we can furthermore predict what γ we would expect and see if the relativistic prediction for time dilation agrees with our experiment.

There is one final refinement to this test. It turns out that there are abundant muons at relativistic velocities produced in cosmic rays! They are produced in the upper atmosphere and decay as they move towards the Earth's surface. Suppose we can assume that the flux of muons from cosmic rays is independent of position on the Earth's surface over a few tens of kilometers, and also independent of time when averaged over a large enough time interval. We can then use cosmic ray muons to do this experiment! This was done in the 1950s by a group at MIT.