Lecture 10 - Applications and course review

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1 Review of electron volts

As we have discussed several times, the nuclear, heavy ion, particle and astrophysics communities make heavy use of the electron volt units. I thought it worth conducting a brief recap on these units before adding one more trick to the bag. An electron volt is the energy gained by an electron in moving between two electrodes between which a potential difference of one volt is applied (the destination electrode is the more positive of the two). It is easy to see why $1\,\mathrm{eV} = 10^{-19}\,\mathrm{J}$. Let us say that the electrodes are separated by a distance d. Then the force on the electron is $F = eV/d$. Then the work done by the electron in moving between the electrodes is $Fd = eV$. Since V is 1 volt, the work done is equal to the numerical value of the electron charge e, in Joules. Therefore $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$.

Recall that since for a particle at rest, the energy $E_R = m_0 c^2$, where m_0 is the rest mass, it is only necessary to specify the rest energy E_R and the rest mass m_0 can always be deduced. Since E_R is almost always a more useful quantity in relativistic problems, it is customary to state that the rest mass of the electron, for example is $0.511\,\mathrm{MeV/c^2}$. This means that the energy of an electron at rest is $0.511\,\mathrm{MeV}$. The c^2 to the right of the units is a reminder that to work out the rest mass, the energy, converted from eV into J should then be divided by c^2 to obtain a rest mass in kilograms.

Recall also that there is also a quantity with energy units directly related to the momentum of a particle, $E_P = |\vec{p}|c$. Indeed, for a massless particle such as a photon, E_P is the energy of a photon having momentum $|\vec{p}|$. For other massive particles, the total energy E is related to the momentum energy E_P and the rest energy E_R by

$$E^2 = E_P^2 + E_R^2. \tag{1}$$

This is the same equation as the energy–momentum–mass relation,

$$E^2 = p^2c^2 + m_0^2c^4. \tag{2}$$

Sometimes it is useful to use electron volt units even in non-relativistic problems. For example, recall that the kinetic energy T of a relativistic particle is

$$T = (\gamma - 1)m_0c^2. \tag{3}$$

However, in the case where one has a particle moving much slower than the speed of light, the kinetic energy can be written as $T_{\text{NON RELATIVISTIC}} = mv^2/2$. Even though this particle is non-relativistic, the kinetic energy can be deduced from the rest energy $E_R = m_0c^2$ and $\beta = v/c$, as follows:

$$T_{\text{NON RELATIVISTIC}} = \frac{1}{2}m_0v^2 = \frac{1}{2}\left(m_0c^2\right)\left(\frac{v^2}{c^2}\right) = \frac{1}{2}E_R\beta^2. \tag{4}$$

So, for example, a particle whose rest mass is $1\,\mathrm{GeV/c^2}$ moving at 1/1000 of the speed of light has a kinetic energy of $500\,\mathrm{eV}$. This method is often easier than converting back to SI units and using $T_{\text{NON RELATIVISTIC}} = mv^2/2$. Similarly for non-relativistic momentum we have

$$\begin{aligned} p_{\text{NON RELATIVISTIC}} &= m_0v \\ E_P = p_{\text{NON RELATIVISTIC}}c &= m_0vc = \left(m_0c^2\right)\left(\tfrac{v}{c}\right) = E_R\beta. \end{aligned} \tag{5}$$

So, for example, the same particle has a momentum of $1\,\mathrm{MeV/c}$.

Throughout the above treatment, I have used the rest mass m_0 of a particle, defined as the energy released were the particle to be converted entirely to energy. I did also discuss the inertial mass m, defined as the ratio of the momentum of a particle to its velocity. Recall that these two masses associated with the same particle are related by $m = \gamma m_0$, and that the modern convention is to adopt the rest mass m_0 into common use, and to use the total energy E in place of the inertial mass, since $E = mc^2$ just as $E_R = m_0c^2$.

2 De Broglie Wavelength

In quantum mechanics, the nature of a scattering event is often best found out by deducing the De Broglie wavelength of an incident particle and comparing it with the dimension of the target. For example, if the target is a nucleus, which typically has a dimension of a few fermi, or a few $\times 10^{-15}$ m, an incoming projectile having a De Broglie wavelength very much shorter than the dimension of the nucleus, if it scatters elastically, will have the scattering resemble the classical collision of two hard spheres. If on the other hand, the De Broglie wavelength of the incoming projectile is of the order of, or longer than, the dimension of the target, the scattering will be much 'softer', and correct analysis will involve quantum mechanics which allows or interference between different paths that the projectile might take when interacting with the target. The De Broglie wavelength λ is given by

$$\lambda = \frac{h}{p}, \tag{6}$$

where p is the momentum of the incoming particle. Once again, although it is an option to convert the momentum to $\mathrm{kg\,m\,s^{-1}}$, there is typically an easier way, since the formula for De Broglie wavelength can be re-written as follows,

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\hbar c}{E_P} \tag{7}$$

Now, a very useful thing to remember to save time when calculating De Broglie wavelengths is that

$$\hbar c = 0.2\,\mathrm{GeV\,fm}. \tag{8}$$

Taking our particle having rest mass $1\,\mathrm{GeV/c^2}$ and velocity 0.001c again, it's momentum energy was 1 MeV. Therefore its De Broglie wavelength is $6.28 \times 0.2[\mathrm{GeV\,fm}]/0.001[\mathrm{GeV}] = 1256\,\mathrm{fm}$, so a collision of this particle with any stationary nucleus would be very soft indeed - more like diffraction of a light beam by a spherical ball than two hard spheres colliding! If the incident particle is relativistic, then the relativistic momentum energy should be calculated using $E_P = \sqrt{E^2 - E_R^2}$; once you have E_P the De Broglie wavelength follows from Equation 7.

For another example, try this one:

Will a 120 keV gamma ray striking a proton have a soft or hard collision with it? For a gamma ray, $E_P = E$, so the momentum energy is 120 keV. The De Broglie wavelength is then $2\pi \times 0.2[\mathrm{GeV\,fm}]/0.000120[\mathrm{GeV}]$, which is several thousand fermi, so the collision with a single proton will be soft. The story would be very different for 100 GeV cosmic ray gammas, for which the De Broglie wavelength is 0.012 fermi; these gammas would undergo hard-sphere elastic collisions with protons.

3 Approximations for highly relativistic particles

It is often true that particles in relativity problems are highly relativistic, meaning that they are moving extremely close to the speed of light. Under such circumstances, we can make approximations that will prove useful for doing problems. Suppose we are told to calculate how much slower than the speed of light a particle with a gamma factor γ is moving, where γ is significantly greater than 1. Let the difference between the particle speed and c be ε. Then we have

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{1}{\sqrt{1-\frac{(c-\varepsilon)^2}{c^2}}} \\
&= \frac{1}{\sqrt{\frac{c^2-(c-\varepsilon)^2}{c^2}}} \\
&\simeq \frac{1}{\sqrt{\frac{c^2-c^2+2c\varepsilon}{c^2}}} \\
&\simeq \sqrt{\frac{c}{2\varepsilon}}.
\end{aligned} \tag{9}$$

This approximation is only valid for very relativistic particles, say for γ greater than 10. However, many for many high energy physics problems this is true of most or all of the particles in the interaction. Here is an example of how to use this:

A high energy proton having total energy E and and a photon are fired from the surface of the Earth towards the moon simultaneously. If the distance to the moon is D find an expression *for the time difference between the arrival of the photon and the arrival of the proton in terms of E, the proton mass M_P, c, and the distance D if $E \gg M_P$.*

The γ factor for the proton is $\gamma = E/(M_P c^2)$. Substituting into Equation 9 we get $c/(2\varepsilon) = E^2/(M_P^2 c^4)$, or $\varepsilon = (M_P^2 c^5)/(2E^2)$. The light travel time to the moon is D/c. The proton travel time is $D/(c-\varepsilon)$. The difference between these times is

$$\frac{D}{c-\varepsilon} - \frac{D}{c} = \frac{D}{c\left(1-\frac{\varepsilon}{c}\right)} - \frac{D}{c} \tag{10}$$

We can use a Binomial expansion to simplify this.

$$\begin{aligned} \frac{D}{c\left(1-\frac{\varepsilon}{c}\right)} - \frac{D}{c} &\simeq \frac{D}{c}\left(1+\frac{\varepsilon}{c}\right) - \frac{D}{c} \\ &\simeq \frac{D\varepsilon}{c^2}. \end{aligned} \tag{11}$$

Therefore the time delay between the arrival of the photon and the proton can be written approximately as

$$\Delta t = \frac{D}{c^2}\frac{c(M_P c^2)^2}{2E^2} = \frac{D}{2c}\left(\frac{M_P c^2}{E}\right)^2 \tag{12}$$

For a 100 GeV proton, assuming a proton rest energy of about 1GeV, and a lunar distance of about 3.8×10^8 m, this time delay is about $(3.8 \times 10^8[\mathrm{m}])/(2 \times 3 \times 10^8[\mathrm{m\,s^{-1}}]) \times (1/100)^2 = 63\mu\mathrm{s}$.

4 Preparation for the exam

Easily the most useful preparation for the exam is practicing examples. The past paper from last year and the practice paper that I produced the first year I took the course, which also has solutions, will be the most useful preparation for the exam this year. In addition, please note that there are certain formulae that you would do well to memorize. Only the Lorentz transformations appear on the front page of the exam paper - all other formulae that you think you might make use of should be committed to memory.

My advice would be to memorise the key results from each of the topics we have studied in detail. These are - time dilation, Lorentz contraction, addition of velocities, the Doppler effect, the relationships between energy, rest mass, and momentum for massless and massive particles, and the relativistic expression for kinetic energy. The Lorentz transformations themselves are written on the front of the exam paper. This means you need to commit about six formulae to memory. This should not be beyond your capabilities.

Regarding derivations, I do not advice the memorizing of long derivations. It is probably useful to be familiar with the basic arguments by which I derived the time dilation, the Lorentz contraction, addition of velocities, and by which I argued that the rest energy is $E = m_0 c^2$.

Other than this, focus on doing the available examples. I am around if you have trouble. I do appreciate an email giving advanced notice of a visit if at all possible. Sometimes the question can be answered by email, saving us all time.

Here are another couple of examples, this time focusing more on relativistic kinematics - the distortions in lengths and times due to motion at velocities near to the speed of light.

5 Example 1

Two identical twins separate, one staying on Earth, at rest, and the other travelling to a star about 20 light years away at a speed of 0.8c. When the moving twin reaches the star, he turns around and comes back again at the same speed. How much older or younger is this twin than his brother when he returns to Earth? Where is the asymmetry between the two brothers?

6 Example 2

A man carrying a pole horizontally runs at a constant speed into a barn whose long axis is parallel to the pole. The barn is 20 metres in length. The rest length of the pole is 25 metres. Once the rear end of the pole enters the barn, the rear door is shut *behind the pole. Although the pole at rest is longer than the barn, and therefore when at rest it won't fit, the runner gets around this difficulty by running so fast that the pole's length is Lorentz contracted, and the pole does fit into the barn. For this problem, consider the runner to be travelling in the $+x$–direction.*

1. At what fraction of c must the runner sprint so that the pole is Lorentz contracted to a length of 15 metres to an observer in the reference frame of the barn? What are the values of the special relativistic β and γ corresponding to this speed?

2. In the (unprimed) reference frame of the barn, considering the event of the rear of the pole entering the barn to have coordinates $(ct = 0, x = 0)$, what are the coordinates of the event of the front of the pole touching the far wall of the barn?

3. Consider a second (primed) observer in the reference frame of the runner. In this frame, calculate the coordinates (ct', x') of:

   (a) The rear of the pole entering the rear of the barn;

   (b) The front of the pole touching the front wall of the barn.

4. In the reference frame of the runner, once the front of the pole hits the front of the barn, how much longer does it take until the rear of the pole enters the barn door? Show that the rear end of the pole doesn't receive information that the front end has hit the far wall until after it has entered the barn. Can the pole be perfectly rigid? Explain your answer.