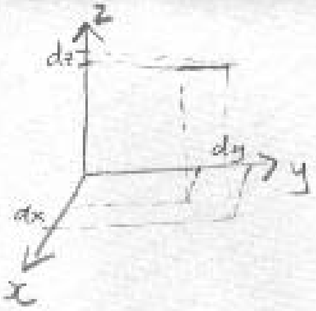
BLACK HOLES

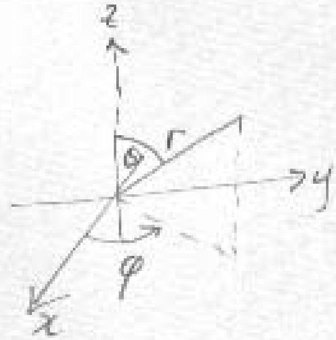
$dl^2 = dx^2 + dy^2 + dz^2$



Invariant under various transformation of the coordinates

* Rotations
* Translations
* Reflections

Different coordinate systems



polar: $dl^2 = \underbrace{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$

looks different, it's just still the same space. Whats different? ~~Unit vec~~ Basis vectors are functions of position

Still the same flat 3d space, but we have chosen a CURVILINEAR coordinate system.

Curved spaces take flat 3d space in polar coordinates, but set $r = R$ constant

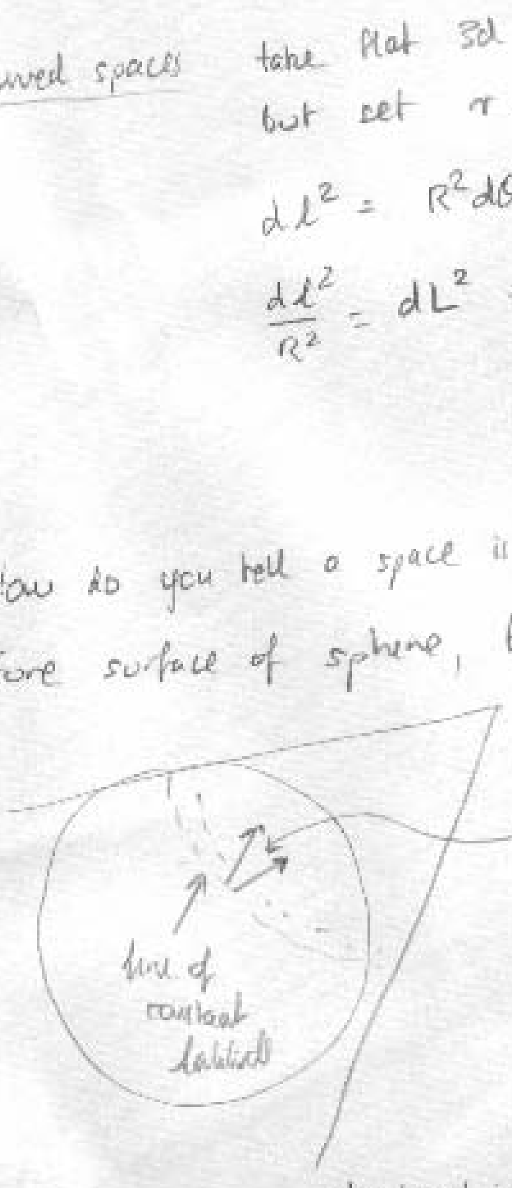
$dl^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2 \leftarrow$ the ul

$\frac{dl^2}{R^2} = dL^2 = \underbrace{d\theta^2 + \sin^2\theta \, d\varphi^2} \leftarrow$

looks different, AND represents a curved space

How do you tell a space is curved? by measuring its curvature

For surface of sphere, there's a nice trick



on return, the arrow no longer points in its original direction: signature of curved space, given a name - the curvature

$R^{\sigma}{}_{\mu\nu\lambda}$

σ ← what component of vector you are measuring

λ ← initial direction of vector

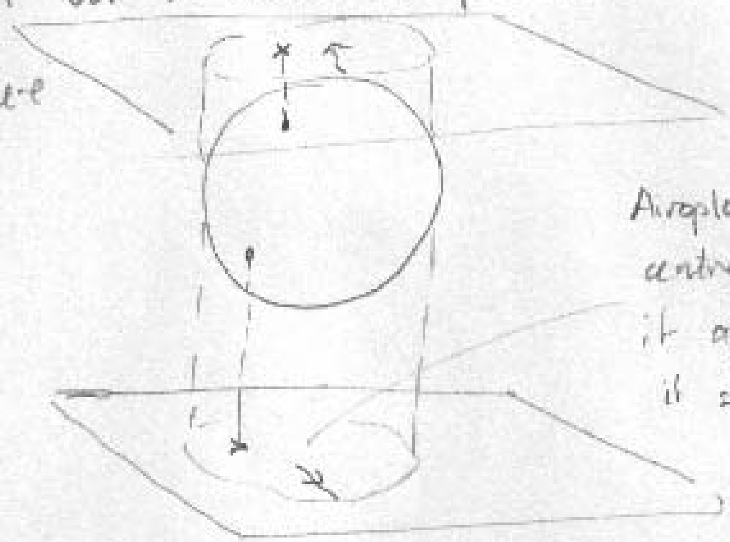
ν ← direction of subsequent

μ ← direction of initial

Can't you just always use flat space coordinates?

Well, perhaps but it leads to problems

Surface of sphere



Aeroplane flies across centre of 'space'; as it approaches the edge, it seems to slow down, the rate of change of its spatial coordinates wrt. the time coordinate drops.

Then it disappears.... then it instantaneously re-appears on the other ~~flat sp~~ coord sheet.

Looks wierd, but the weirdness just reflects our point of view.

Space-time Einstein taught us how to find quantities that are invariant under rotations, translations, reflections, AND boosts in the velocity of the observer.

In a flat space-time with a speed of light c consider 2 events separated in space by (dx, dy, dz) and in time by dt

then $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ is invariant under these transformations.

ds^2 is called the interval between the two events the Metric in a flat spacetime can be written . . well we just wrote it.

But we may also want to use spherical coordinates for the spatial bit. Then we get

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

This is a flat spacetime. Can we make spacetime curved?
Yes. and Einstein told us how.

Curved spacetime is a consequence of the local presence of non-zero energy density

Large masses close by cause ~~to~~ non ~~to~~ zero local energy density

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Related to the curvature

newtons gravitational constant

Energy-momentum tensor

Einstein's equations

Very hard to solve. But schwarzschild found a solution (for $G_{\mu\nu}$) ; ~~from it we can write~~ we can write the solution in terms of a metric.

$$ds^2 = -c^2dt^2\left(1 - \frac{2GM(r)}{rc^2}\right) + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

The physics $M(r)$: mass enclosed within radius r

G newton's gravitational constant

c speed of light

What happens as $r \to \frac{2GM(r)}{c^2}$?

In otherwords, what if the ~~mass is so~~ density is sufficiently great that for some radius r_s, =

$$M(r_s) \geq \frac{r_s c^2}{2G}$$

Well, infinities appear in the metric as you approach r_s.

r_s is called the Event Horizon

1d flat space photons move at speed of light.

$$ds^2 = -c^2dt^2 + dr^2$$

$$\text{but } dr = cdt \Rightarrow ds = 0$$

for photon, $ds = 0$ along path

photons move on null geodesics, in free fall

so, since ds is an invariant, $ds = 0$ along the path of a photon near a black hole.

$$c^2 dt^2 \left(1 - \frac{r_s}{r}\right) = \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)}$$

$r_s = \frac{2GM}{c^2}$

all mass within r_s
so M indep. of r

$$\Rightarrow \quad c^2 dt^2 = \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)^2}$$

$$\pm c\,dt = \frac{dr}{\left(1 - \frac{r_s}{r}\right)}$$

let $u = \frac{r}{r_s}$ $\quad dr = r_s\, du$

$$\pm c \int dt = r_s \int \frac{du}{1 - \frac{1}{u}}$$

$$\pm c \int dt = r_s \int \frac{u\, du}{u - 1}$$

$$= r_s \int \frac{u - 1 + 1}{u - 1}\, du$$

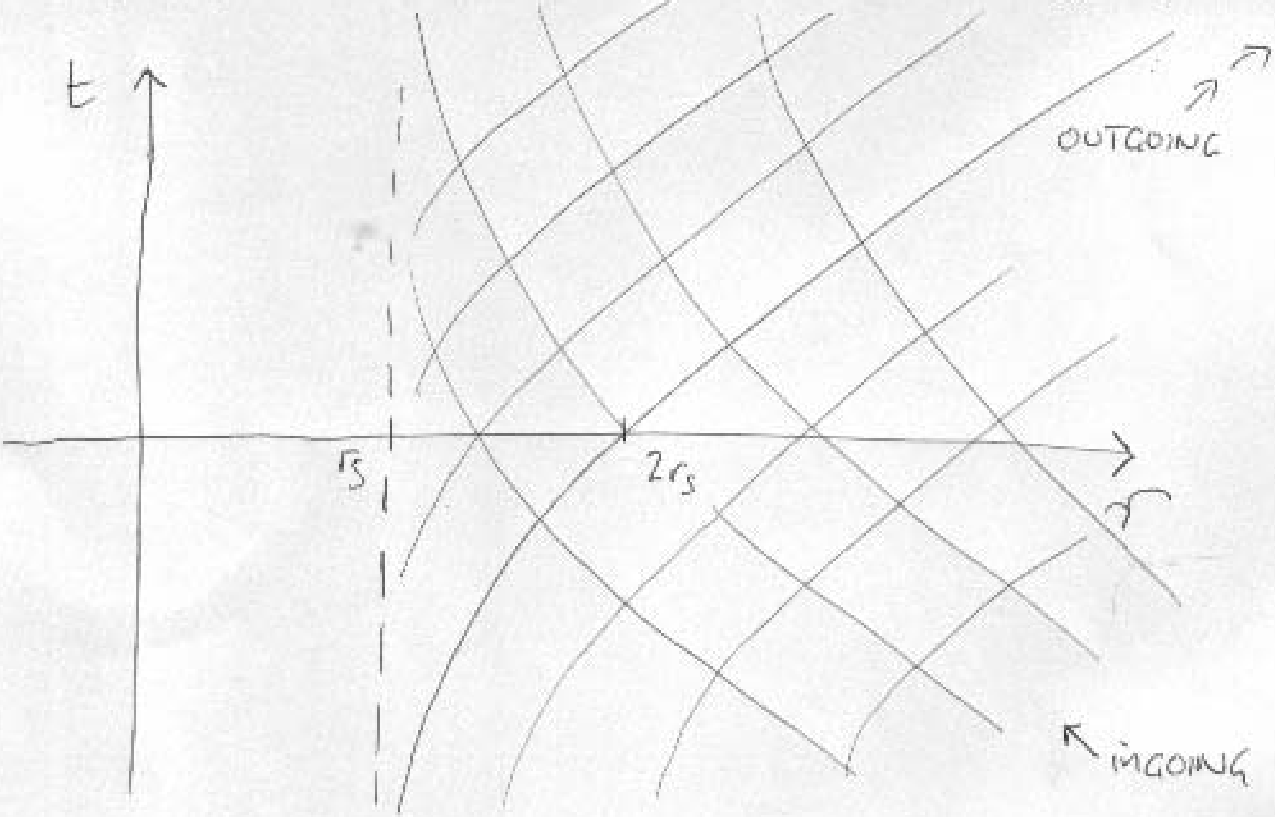
$$= r_s \int du + r_s \int \frac{du}{u - 1}$$

$$= r_s \int du + r_s \ln u - 1$$

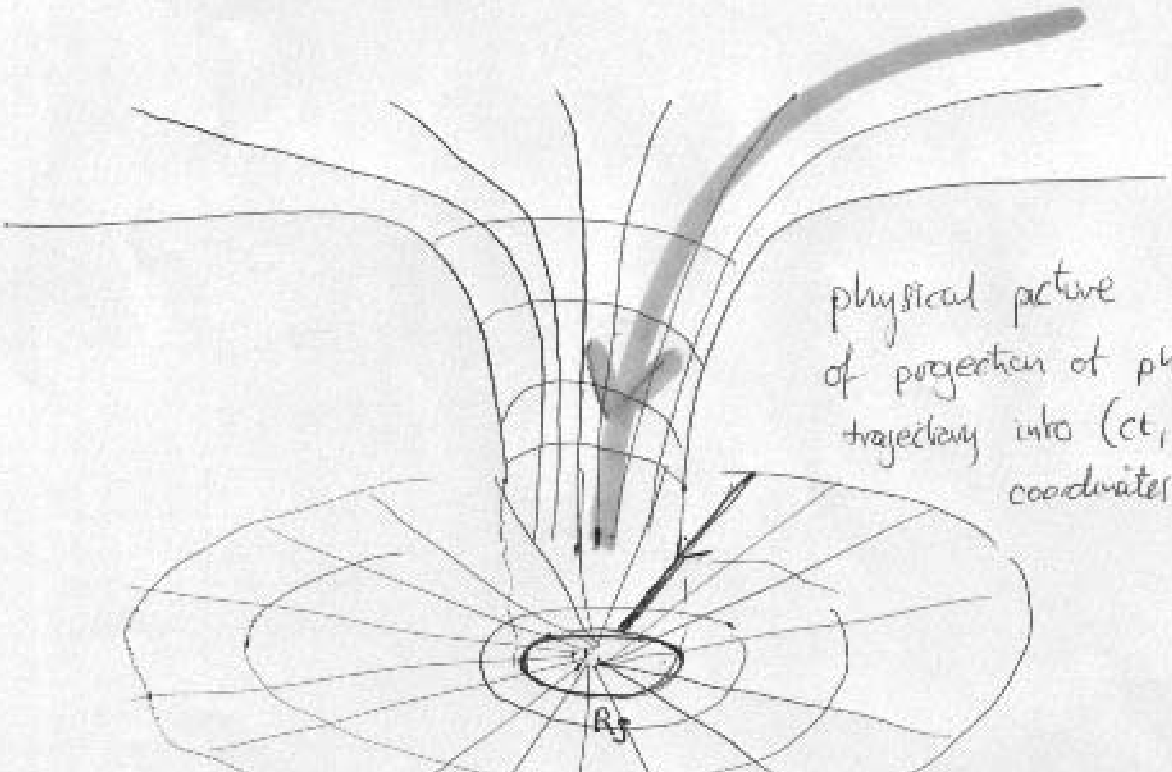
$$= r_s u + r_s \ln u - 1$$

$$\pm c t = A + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$

Solutions for a Photon on a radial trajectory



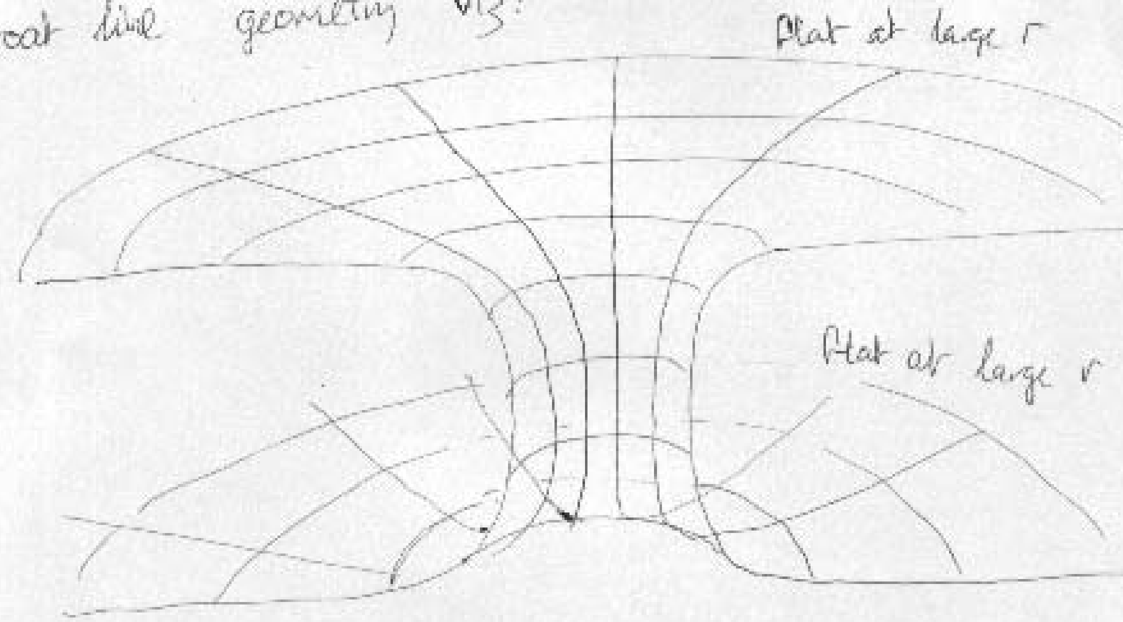
the COORDINATE rate of change $\frac{dr}{dt}$ ~~appears~~ decreases as an incoming photon approaches the Schwarzschild radius. This is OK - it's just a consequence of the highly curved nature of the spacetime in conjunction with the coordinate system (ct, r, θ, φ)



physical picture of projection of photon trajectory into (ct, r, θ, φ) coordinates.

Wormholes

If you use a more appropriate coordinate system, called Kruskal coordinate, the whole of r, θ, φ space only takes up half the coordinate system — including a flat space region far from the schwarzschild radius and the highly curved region closer in. The other half of the space can be discarded, or taken seriously. If you take it seriously, it leads to a second asymptotically flat region, with the two regions connected by a curved 'throat like' geometry viz:



in the schwarzschild geometry it is impossible for anything to travel from one asymptotically flat region to the other, but it is possible for light that has crossed the schwarzschild radius from one flat region to intersect with an observer who has crossed the schwarzschild radius from the other flat region. So if you ever fall in to a black hole — look behind you!