DISTANCES AT MODERATE REDSHIFT

INTRODUCTION

Measuring—and even defining—distances at redshifts $z > 0.1$ is a non-trivial task. Students doing PHY306 *Introduction to Cosmology* will encounter this problem in their course; this is a brief summary for students who have not chosen this option.

From the perspective of distance measurement there are essentially three different situations:

1. *Low redshift, $z < 0.05$*. In this region the distance is more-or-less independent (to about 10%) of the cosmological model or the method of measurement. It is safe to use $cz = H_0d$, where $H_0 = 70–75$ km s^{-1} Mpc^{-1}, to calculate extragalactic distances in this redshift range.
2. *Moderate redshift, $0.05 < z < 1.0$.* In this range it is *not* safe to use $cz = H_0d$, and the distance you are measuring depends on the method used to measure it. However, there are useful approximate formulae which will give a reasonably accurate value for distances in this range.
3. *High redshift, $z > 1.0$*. In this range you need an explicit cosmological model to derive the distance, and for the current benchmark model ($\Omega_\Lambda \approx 0.73$, $\Omega_m \approx 0.27$) the calculation is not trivial.

This note concerns the second regime. For studies of dark matter, the main relevance of this redshift range is for strong and weak gravitational lensing (gravitational microlensing, e.g. in searches for Galactic MACHOs, is a local phenomenon).

DISTANCE MEASUREMENT

The 'natural' interpretation of a distance measurement is that it is the result you would get if you stretched a very long ruler between the target object and the observer and read off the answer from the ruler scale: that is, it's the separation between two objects considered at the same time. This is often called the *proper distance* between the two objects. Unfortunately, the proper distance is impossible to measure in astronomy, because light takes a finite time to travel from the source to the observer. In practice, astronomical distance measurements take one of two forms:

1. *Standard candle methods*. In these methods, we measure the flux f (energy per unit area per unit time) received from an object whose total luminosity we believe we know. Assuming that at distance d this luminosity is spread out over a sphere of area $4\pi d^2$ gives us
$$f = \frac{L}{4\pi d^2}$$
(or, equivalently, $m - M = 5\log(d/10)$).

   Distances measured in this way are called *luminosity distances* and given the symbol d_L.

   The distances measured using Type Ia supernovae are luminosity distances.

2. *Standard ruler methods.* In these methods, we measure the *angular* size of something whose *linear* size we think we know. Angular sizes in astronomy are always small, so we use the small angle approximation to get

$$\theta = \ell/d,$$

where θ is the angular size, ℓ is the linear size, and d is the distance. Distances measured in this way are called *angular diameter distances* and denoted d_A.

The distances measured using gravitational lensing are angular diameter distances.

The problem is that these distances are not equivalent to each other, and neither of them is equivalent to the proper distance. This is explained fully in PHY306, but the basic physics is fairly simple:

- In an expanding universe, the photons arriving from a distant source are redshifted down in energy, and also arrive at a lower rate compared to what you would expect in a non-expanding model. Therefore the received flux is less than we might expect, and so the luminosity distance is *greater* than the proper distance.
- On the other hand, the transverse distance corresponding to a given linear size expands as the universe expands, so the measured angular size is larger than we might expect in a non-expanding cosmology. Therefore the angular diameter distance is *less* than the proper distance.

It turns out that for any expanding cosmological model, $d_L = (1+z)^2 d_A$, where z is the redshift. For models with flat geometry, it is also true that $d_L = (1+z)d_P$, where d_P is the proper distance. As measurements of the cosmic microwave background indicate that our universe *is* flat (within experimental error), it is appropriate to use this formula in interpreting real data.

APPROXIMATE FORMULA FOR LUMINOSITY DISTANCE

Hubble's law, $cz = H_0 d$, is valid at low redshift. At higher redshift, we expect deviations from the simple law, because the value of H changes with time. The deviations can be parameterised using a power series expansion in redshift, which gives the result

$$H_0 d_L \simeq cz\left(1 + \tfrac{1}{2}(1 - q_0)z\right)$$

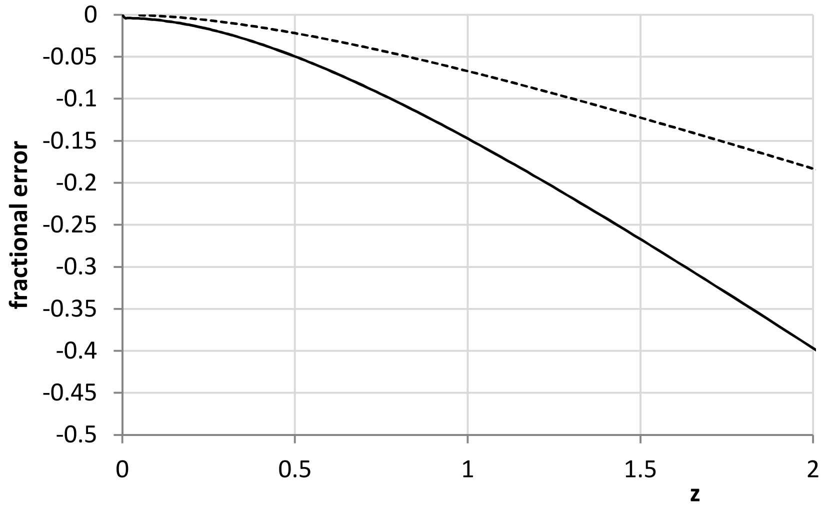
for luminosity distance. The *deceleration parameter* $q_0 = \frac{1}{2}\Omega_{\mathrm{m}} - \Omega_\Lambda$ for a flat geometry and negligible radiation density; for the standard benchmark values $q_0 = -0.60$. (This definition was introduced in the days when the cosmological constant ('dark energy') was confidently expected to be zero, which implies that the expansion rate is always decreasing with time; therefore q_0 was defined with a built-in minus sign so that it would be positive for decelerating expansion. The fact that it is numerically negative indicates that the expansion rate is in fact increasing.)

Because this is the result of a power series expansion, it is only approximately correct. For luminosity distances, it is good within 10% out to redshifts of about 0.75 or so (the error at redshift 1 is about 15%) for the standard benchmark cosmology (it is better than this for a matter-only cosmology, without dark energy). Since modern cosmology often involves redshifts significantly greater than this, the q_0 formalism is not much used: if you read a modern research paper, you will find deviations from the Hubble law described in terms of deduced values of Ω_{m} and Ω_Λ, not in terms of q_0. However, it is good enough for the distances involved in most studies of dark matter in galaxy clusters.

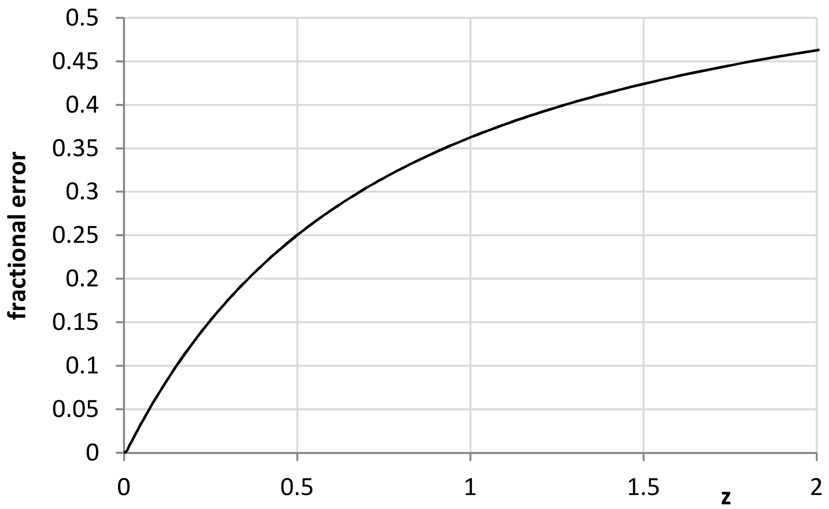
Although there are similar approximate formulae for proper distance and angular diameter distance, they are not as good: it is better to calculate the luminosity distance from the formula and then use $d_P = d_L/(1+z)$ and $d_A = d_L/(1+z)^2$ to calculate the other distances.

QUALITY OF APPROXIMATION

The plot below shows the fractional error $(d - d_{q_0})/d$ for the formula above (since the conversions to proper distance and angular diameter distance are exact, the fractional error is the same for all the definitions of distance). The solid line is for the standard benchmark model, in which $\Omega_m = 0.267$; the dashed line is for a universe in which $\Omega_m = 1$ (the old “standard cold dark matter” model of the 1980s and 90s). Note that the approximation always overestimates the true distance, and that it is better for the matter-only universe than it is when dark energy is added.



For comparison, the plot below shows the fractional error that arises if we use the simple Hubble law, $cz = H_0 d$, to calculate the distance in the standard benchmark cosmology. In this redshift range, the error produced by doing this is much larger than it is if we use the q_0 formula (25% at $z = 0.5$ instead of 5%). This is compared to luminosity distance; the error if we compare to angular diameter distance is *huge* (68% at $z = 0.5$!) and has the opposite sign.



SUMMARY

If you are dealing with redshift $z > 0.05$ or so (0.1 if you’re not too fussy), you need to specify whether you are measuring a luminosity distance or an angular diameter distance, and you need to take account of deviation from the linear Hubble law. The approximate formula for luminosity distance, $H_0 d_L \simeq cz\left(1 + \frac{1}{2}(1 - q_0)z\right)$, where $q_0 = \frac{1}{2}\Omega_\mathrm{m} - \Omega_\Lambda$, may be used out to redshifts of order 0.5 (for 5% precision) to 1 (for 15% precision)—beyond that, explicit cosmological model calculations are necessary.

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