Modern cosmology 3:
The Growth of Structure

- Growth of structure in an expanding universe
- The Jeans length
- Dark matter
- Large scale structure simulations
  - effect of cosmological parameters
- Large scale structure data
  - galaxy surveys
  - cosmic microwave background

Growth of structure

- Consider sphere of radius $R$ and mass $M$
  - add small amount of mass so that $\rho = \bar{\rho}(1 + \delta(t))$
  - gravitational acceleration at edge of sphere is
    $$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi}{3}G\bar{\rho}R(1 + \delta(t))$$
  - since $\rho(t) = \rho_0a^3$, conservation of mass gives $R(t) \propto a(t)[1 + \delta(t)]^{-1/3}$
  - differentiating this twice gives
    $$\frac{\ddot{R}}{R} \approx \frac{\ddot{a}}{a} - \frac{\delta}{3} = -\frac{2}{3} \frac{\dot{a}}{a} \delta$$
**Growth of structure**

- Now have two equations for $\dddot{R}/R$
  
  $\frac{\dddot{R}}{R} = -\frac{GM}{R^3} = -\frac{4\pi}{3} \rho \delta(t)$

  $\frac{\dddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{2}{3} \frac{\ddot{a}}{a} \delta$

- Combine to get
  
  $-\frac{4}{3} \pi G \rho - \frac{4}{3} \pi G \rho \delta = \frac{\ddot{a}}{a} - \frac{2}{3} \frac{\ddot{a}}{a} \delta$

- Subtract off $\delta = 0$ case to get
  
  $\ddot{\delta} + 2H \dot{\delta} - \frac{3}{2} \Omega_m H^2 \delta = 0$

  since $\Omega_m = \frac{8\pi G \rho}{3H^2}$

**Radiation dominated**

\[ \ddot{\delta} + 2H \dot{\delta} - \frac{3}{2} \Omega_m H^2 \delta = 0 \]

- $H = 1/2t$ and $\Omega_r \gg \Omega_m$, so $\ddot{\delta} + \frac{1}{t} \dot{\delta} \approx 0$

  - the solution of this is $\delta(t) = C_1 + C_2 \ln t$
  - any overdensity grows only logarithmically with time
**Inflation/Λ dominated**

\[ \ddot{\delta} + 2H \dot{\delta} - \frac{3}{2} \Omega_m H^2 \dot{\delta} = 0 \]

- \( H = H_\Lambda \) and \( \Omega_\Lambda \gg \Omega_m \), so \( \ddot{\delta} + 2H_\Lambda \dot{\delta} = 0 \)

- the solution of this is \( \delta(t) = C_1 + C_2 \exp(-2H_\Lambda t) \)
- overdensity does not grow at all

**Matter dominated**

\[ \ddot{\delta} + 2H \dot{\delta} - \frac{3}{2} \Omega_m H^2 \dot{\delta} = 0 \]

- \( H = 2/3t \), so if \( \Omega_m = 1 \) \( \ddot{\delta} + \frac{4}{3t} \dot{\delta} - \frac{2}{3t^2} \delta = 0 \)
  - try a power law solution \( \delta(t) = C t^n \)
  - get \( 3n^2 + n - 2 = 0 \)
  - \( n = -1 \) or \( n = \frac{2}{3} \)
  - \( \delta(t) = C_1 t^{-1} + C_2 t^{2/3} \)
  - overdensity does grow
  - growth \( \propto t^{2/3} \propto a \)
The Jeans length

- Astrophysical objects are stabilised against collapse by pressure forces
  - Pressure forces travel at the speed of sound, \( c_s = c \left( \frac{\partial P}{\partial \varepsilon} \right)^{1/2} = w^{1/2} c \)
  - Collapse occurs if the size of the object is smaller than the Jeans length
    \[
    \ell_J = c_s \sqrt{\frac{\pi}{G \varepsilon}} = c^2 \sqrt{\frac{w \pi}{G \varepsilon}}
    \]
    where \( \varepsilon \) is the mean energy density
      - (exact numerical factor depends on details of derivation)
      - Can also be expressed as Jeans mass

We know that for a flat matter-dominated universe
\[
H = \left( \frac{8\pi G \varepsilon}{3c^2} \right)^{1/2}
\]
- So \( \ell_J = c^2 \sqrt{\frac{w \pi}{G \varepsilon}} = \sqrt{w} \times 2\pi \frac{2}{3} \frac{c}{H} \)

- At last scattering \( H^2 = \Omega_{m0} H_0^2 / a^3 \), giving \( c/H \sim 0.2 \) Mpc for \( \Omega_{m0} = 0.3 \)
  - So for radiation \( \ell_J \sim 0.59 \) Mpc
    - ~ horizon size, so nothing below horizon size collapses
  - For matter \( w = kT/\mu c^2 = 2.3 \times 10^{-10} \), so \( \ell_J \sim 15 \) pc
    - Equivalent to current size ~ 15 kpc, a small galaxy
Conclusions
(radiation-dominated)

- Below Jeans length (~horizon size), nothing collapses
  ▶ supported by pressure
- Above Jeans length
  ▶ fluctuations in matter grow only logarithmically
  ▶ fluctuations in radiation grow \( \propto a^2 \)
    ▶ derivation similar to p3 gives \( \ddot{\delta} + 2H\dot{\delta} - 4\Omega\gamma H^2\delta = 0 \)
    ▶ solve as in p6 to get \( \delta \propto t \)
    ▶ but \( \ell_J \propto a^2 \), so as universe expands fluctuations will “enter the horizon” and stop growing

Conclusions
(matter-dominated)

- Before matter-radiation decoupling, photons and baryons form a single gas
  ▶ this is stabilised by radiation pressure
  ▶ density perturbations cannot grow
- After decoupling, baryon gas has galaxy-scale Jeans length
  ▶ galaxy-sized objects start to grow \( \propto a \)
  ▶ but it turns out that this is too slow to produce structures we see
  ▶ (note that our derivation is only valid for small \( \delta \))
**Dark Matter**

- In fact we know $\Omega_{\text{baryon}} \sim 0.04$ from light element abundances, whereas $\Omega_m \sim 0.25$ from cluster X-ray data
  - most matter is not made of baryons
    - (non-baryonic dark matter)
  - this does not couple to photons
  - dark matter structures can grow from time of matter-radiation equality ($z \sim 3200$)
  - this is much more promising

**Structure formation with dark matter**

- Assume dark matter consists of a weakly interacting particle with mass $m_{\text{dm}}$
  - such a particle will be relativistic until the temperature drops to $T_{\text{dm}} \sim m_{\text{dm}}c^2/3k$
  - if this is less than $T_{\text{rm}} \sim 10^4$ K, the dark matter is hot; if more, it is cold
    - for hot dark matter the minimum scale for collapse is given by $[ct_{\text{dm}} = ct_{\text{rm}}(T_{\text{dm}}/T_{\text{rm}})^2] \times (T_{\text{dm}}/2.74)$
    - for cold dark matter it is dwarf galaxy sized
Conclusions
(dark matter)

- Hot dark matter candidate: neutrino with mass ~2 eV/c²
  - $T_{dm} = 7750 \text{ K}$
  - minimum length scale (for $t_{rm} \sim 50000 \text{ yr}$)
    - ~70 Mpc
    - supercluster sized
- Large structures form first

- Cold dark matter candidate: WIMP with mass ~100 GeV/c²
  - $T_{dm} = 4 \times 10^{14} \text{ K}$
  - minimum length scale set by Jeans length for matter
  - small galaxy sized
- Small structures form first