**Cosmological models**

- Cosmological distances
- Single component universes
  - radiation only
  - matter only
  - curvature only
  - Λ only
- Multi component universes

**Cosmological distances**

- Proper distance between origin and object:
  - $ds^2 = -c^2 dt^2 + a^2(t) [dr^2 + x(r)^2 dΩ^2]$ (R-W metric)
  - $d_p(t) = a(t) \int dr = a(t) r$ ($d_p$ is not a comoving distance)
  - but we know $r = c \int_{t_e}^{t_o} \frac{dt}{a(t)}$ for light emitted at $t_e$ and observed at $t_o$
  - therefore the proper distance to an object at time $t_0$ is
    \[ d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \]
  - if $t_e = 0$ we call this the horizon distance – it’s the furthest we can currently see
Cosmological models

\[ \dot{a}(t)^2 = \frac{8\pi G}{3c^2} \left( \frac{\varepsilon_{r0}}{a(t)^2} + \frac{\varepsilon_{m0}}{a(t)} \right) - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a(t)^2 \]

- Different components of energy density just add:
  - note different \( a \) dependence
  - at small \( a \), radiation must dominate
  - matter takes over when \( a > \varepsilon_{r0}/\varepsilon_{m0} \)
  - at large \( a \), cosmological constant dominates if it exists
- Therefore sensible to consider single components

Radiation only

\[ a \, da = \sqrt{\frac{8\pi G \varepsilon_{r0}}{3c^2}} \, dt \]

- \( a = (t/t_0)^{1/2} \) and \( \varepsilon_r \propto t^{-2} \)
- age of universe: \( \ln a = \frac{1}{2}(\ln t - \ln t_0) \rightarrow H = 1/2t \)
  - so \( t_0 = 1/2H_0 \)
- proper distance:
  \[ d_p(t_0) = ct_0^{1/2} \int_{t_c}^{t_0} \frac{dt}{\sqrt{t}} = 2ct_0 \left( 1 - \frac{t_c}{t_0} \right) = \frac{c}{H_0} \left( \frac{z}{1+z} \right) \]
  \[ d_p(t_c) = d_p(t_0)/(1+z) \]
**Matter only**

\[ \sqrt{a} \, da = \sqrt{\frac{8\pi G \varepsilon_m}{3c^2}} \, dt \]

- \( a = \left(\frac{t}{t_0}\right)^{2/3} \) and \( \varepsilon_m \propto t^{-2} \)
- age of universe: \( \ln a = \frac{2}{3}(\ln t - \ln t_0) \rightarrow H = \frac{2}{3t} \)
  - so \( t_0 = \frac{2}{3}H_0 \)
- proper distance:
  \[
  d_p(t) = ct_0^{2/3} \int_{t_0}^{t} \frac{dt}{t^{2/3}} = 3ct_0 \left(1 - \left(\frac{t}{t_0}\right)^{1/3}\right) = \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right)
  \]
  \[
  d_p(t_e) = d_p(t_0)/(1+z)
  \]

**Curvature only**

\[ da = \sqrt{\frac{k c^2}{R^2}} \, dt \]

- if \( k = 0 \), \( a = \) constant: flat, static, empty universe
- if \( k = -1 \), \( a \propto t \): universe expands at constant speed
  - Milne model
  - age = \( 1/H_0 \)
  - proper distance \( d_p(t_0) = ct_0 \ln(1+z) \)
- \( k = +1 \) does not produce a physically viable model
**Λ only**

\[
\frac{da}{a} = \sqrt{\frac{\Lambda}{3}} \, dt = H_0 \, dt
\]

- \( a = \exp[H_0(t-t_0)] \): universe expands exponentially
  - de Sitter model
  - infinitely old: \( a \to 0 \) only as \( t \to -\infty \)
  - proper distance \( d_p(t_0) = cz/H_0 \)
- this is a “Steady State” universe which always looks the same

**Single component universes**

![Graphs and diagrams showing expansion and proper distance for different components (Radiation, Matter, Curvature, and Lambda).](image)
**Multi-component universes**

\[ \dot{a}(t)^2 = H_0^2 \left( \frac{\Omega_{r0}}{a(t)^2} + \frac{\Omega_{m0}}{a(t)} + (1 - \Omega_{r0} - \Omega_{m0} - \Omega_{\Lambda0}) + \Omega_{\Lambda0} a(t)^2 \right) \]

- “This is not a user-friendly integral” (Ryden)
  - fortunately at different times different components will dominate
    - best current values: \( \Omega_{m0} = 0.27, \Omega_{\Lambda0} = 0.73, \Omega_{r0} = 8.4 \times 10^{-5} \)
    - matter-radiation equality at \( a = \Omega_{r0}/\Omega_{m0} = 0.0003 \)
    - matter-\( \Lambda \) equality at \( a = (\Omega_{m0}/\Omega_{\Lambda0})^{1/3} = 0.72 \)
  - at any given time can usually use single-component model

**Example: matter + \( \Lambda \)**

![Graph showing expansion rate for matter and \( \Lambda \) dominated models.](image)
**Example: matter + \( \Lambda \)**

![Graph showing the relationship between \( a(t) \) and \( H_0(t-t_0) \).]

**State of Play: theory**

- Friedmann model plus cosmological constant can describe wide variety of behaviour
  - expanding, recollapsing or static
  - also “bouncing” and “loitering” models
  - this technology all available in 1920s
- However, models have free parameters
  - \( H_0, \Omega_m, \Omega_r, \Omega_\Lambda \)
  - need to determine these to see what model predicts for our universe