Relativistic models

- **General relativity allows one to calculate the behaviour of space-time in the presence of mass-energy** ***→PHY314***
- **To see basic behaviour**
  - **use Newtonian approximation where possible**
  - **assume some GR results**

The Friedmann equation

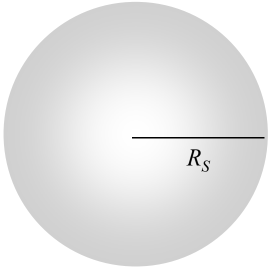
- **Sphere of mass *M*, radius R_S, expanding or contracting**

$$\ddot{R}_S = -\frac{GM}{R_S^2}$$

$$\tfrac{1}{2}\dot{R}_S^2 = \frac{GM}{R_S} + U = \frac{4\pi G\rho}{3} R_S^2 + U$$

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \tfrac{8}{3}\pi G\rho(t) + \frac{2U}{r_S^2 a(t)^2}$$

where $R_S = a(t)\, r_S$ and r_S is the radius of the sphere now



The Friedmann equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \tfrac{8}{3}\pi G\rho(t) + \frac{2U}{r_S^2 a(t)^2}$$

- If $U > 0$
  - RHS always positive
  - universe expands forever
- If $U = 0$
  - RHS → 0 as $t \to \infty$
  - universe expands at ever-decreasing rate
- If $U < 0$
  - $\rho(t) \propto (r_S\, a)^{-3}$
  - U-term $\propto (r_S\, a)^{-2}$
  - at $a = -4\pi G\rho_0/3Ur_S$ expansion reverses
  - universe headed for a Big Crunch

The Friedmann equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \tfrac{8}{3}\pi G\frac{\varepsilon(t)}{c^2} - \frac{kc^2}{R_0^2 a(t)^2}$$

relativistic form

- Replace mass density by energy density
  - $\varepsilon = \rho c^2$
  - contributions from photons etc. as well as matter
- Replace U by curvature
  - $k = +1$ corresponds to $U < 0$ (Big Crunch)
  - R_0 corresponds to present radius of curvature

Ω

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \tfrac{8}{3}\pi G\frac{\varepsilon(t)}{c^2} - \frac{kc^2}{R_0^2 a(t)^2}$$

- **Energy density for *k*=0**
  - **$\varepsilon_c = 3c^2H^2/8\pi G$**
  - **define $\Omega \equiv \varepsilon(t)/\varepsilon_c(t)$**
  - **then**

$$H(t)^2(1-\Omega) = -\frac{kc^2}{R_0^2 a(t)^2}$$

- **H^2, R^2, a^2, c^2 all positive**
  - **$\Omega = 1 \leftrightarrow k = 0$**
  - **$\Omega < 1 \leftrightarrow k < 0$**
  - **$\Omega > 1 \leftrightarrow k > 0$**

Λ

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \tfrac{8}{3}\pi G\frac{\varepsilon(t)}{c^2} - \frac{kc^2}{R_0^2 a(t)^2} + \frac{\Lambda}{3}$$

- **Friedmann equation (like Newton) not static**
  - **Einstein believed Universe is static (pre-Hubble)**
  - **introduced cosmological constant Λ to allow this**
    - basically an integration constant in Einstein's equations
    - can also be expressed as $\Omega_\Lambda = \Lambda/3H^2$; then usually call density parameter Ω_m to distinguish the two

$$H(t)^2(1-\Omega_m-\Omega_\Lambda) = -\frac{kc^2}{R_0^2 a(t)^2}$$

The fluid equation

- **Friedmann equation has two unknowns: ε (or ρ) and a**
  - **need another equation**
  - **try thermodynamics:** $\overset{\text{heat supplied}}{\mathbf{d}Q} = \overset{\text{change in internal energy}}{\mathbf{d}E} + \overset{\text{work done}}{P\,\mathbf{d}V}$
    - energy in volume V is $E = \varepsilon V$; $dE = V d\varepsilon + \varepsilon dV$
    - $V \propto a^3$ so $dV/V = 3da/a$
    - $dQ = 0$ for expansion of universe

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

the fluid equation

The acceleration equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \tfrac{8}{3}\pi G\frac{\varepsilon(t)}{c^2} - \frac{kc^2}{R_0^2 a(t)^2}$$

- **Multiply by a^2 and differentiate:**

$$2\ddot{a}\dot{a} = \frac{8\pi G}{3c^2}\left(2a\dot{a}\varepsilon + \dot{\varepsilon}a^2\right) = \frac{8\pi G\dot{a}}{3c^2}\left(2a\varepsilon - 3(\varepsilon + P)a\right)$$

- **Simplify:**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

Always deceleration unless pressure is negative!

The equation of state

- **Unfortunately we have now introduced a third variable, the pressure P!**
  - **need to relate P and ε (or ρ)**
    - this relation depends on the substance under consideration
    - it is called the equation of state of the substance
  - **some useful equations of state:**
    - non-relativistic gas: $P = nk_BT = \rho k_BT/\mu$ where μ is particle mass
      - since $3k_BT = \mu\langle v^2\rangle$, we have $P/\varepsilon = \langle v^2\rangle/3c^2 << 1$, i.e. $P_m \approx 0$
    - radiation (ultra-relativistic): $P/\varepsilon = ⅓$ so $P_r = ⅓\,\varepsilon_r$
    - Λ: as its energy density is constant with time, $P_\Lambda = -\varepsilon_\Lambda$
      - this gives acceleration, since $\varepsilon + 3P < 0$

The fluid equation revisited

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}\left(\varepsilon + P\right) = 0$$

- **Radiation**
  - $P_r = ⅓\,\varepsilon_r \rightarrow \frac{\dot{\varepsilon}}{\varepsilon} = -4\frac{\dot{a}}{a}$
  - $\varepsilon_r \propto a^{-4}$
- **Nonrelativistic matter**
  - $P_m \approx 0 \rightarrow \frac{\dot{\varepsilon}}{\varepsilon} = -3\frac{\dot{a}}{a}$
  - $\varepsilon_m \propto a^{-3}$
- **Λ**
  - $P_\Lambda = -\,\varepsilon_\Lambda \rightarrow \dot{\varepsilon} = 0$
  - ε_Λ = constant
- **More general form**
  - $P = w\varepsilon \rightarrow \frac{\dot{\varepsilon}}{\varepsilon} = -3(1+w)\frac{\dot{a}}{a}$
  - $\varepsilon \propto a^{-3(w+1)}$

Cosmological models

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \tfrac{8}{3}\pi G\frac{\varepsilon(t)}{c^2} - \frac{kc^2}{R_0^2 a(t)^2}$$

- **Can now substitute in**
  - $\varepsilon = \varepsilon_0 a^{-3}$ **(matter),**
  - $\varepsilon = \varepsilon_0 a^{-4}$ **(radiation), or**
  - $\varepsilon = \varepsilon_0$ **(Λ)**

  in real life a combination of all three!

  to get a differential equation for *a*(*t*)
- **First try this with single components…**