Problem Solving

Calculations

Using expansions

Checks and estimates

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Calculations

- Numerical value: last step in solving a problem
  - usually doesn't cost many exam marks if you get it wrong
    - provided method and logic correct
  - but obviously crucial in real-world applications
    - underestimating stresses on an aircraft wing by factor of 10 clearly not a good idea
- Usual rules apply
  - Know what you're doing
  - Simplify if possible
  - Check answer
    - "calculator typos" common cause of lost marks in exams

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Expansions and Approximations

- Solutions to many problems rely on series expansions
  - e.g. small angle approximations, $\sin\theta \approx \theta$, $\cos\theta \approx 1 - ½\theta^2$
- Know the most useful ones
  - $\sin\theta$, $\cos\theta$, $(1 + x)^n \approx 1 + nx$ (binomial expansion)
    - they are on standard constants sheet if you forget
    - but often needed in derivations, so better to know them
- Be aware of any limitations
  - $\sin\theta$, $\cos\theta$ need θ in radians
- When to use in calculations
  - when required by derivation/definition
  - when they introduce simplifications
  - when you might otherwise have problems with precision

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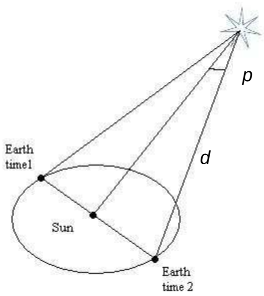
Example 1: Definition



- The parallax of Sirius is 0.379 arc seconds. What is its distance in parsecs?
  - The definition of the parsec is "the distance at which a star would have a parallax of one arc second" (and angles in arc seconds are always small)
    - so d(pc) = 1/p(arcsec) by definition, hence d = 2.64 pc
  - We *could*:
    - convert 0.379" to degrees: 0.0001053°
    - take the sine of that: 1.84×10^{-6}
    - look up the distance to the Sun in metres: 1.50×10^{11} m
    - write $d = 1.50 \times 10^{11} / 1.84 \times 10^{-6} = 8.1 \times 10^{16}$ m
    - look up the conversion between parsecs and metres: 1 pc = 3.09×10^{16} m
    - and convert d to pc: d = 8.1/3.09 = 2.64 pc
    - Same answer, six times more work!

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Example 2: Simplification

- If you are on a boat in mid-ocean, how far away is the horizon? Assume that when standing on the deck your eye level is 10 m above the water surface, and take the Earth to be a sphere of radius 6370 km.
  - You want the short side of a right-angled triangle whose other two sides are 6370 km and (6370km + 10 m).
    - You can do this by Pythagoras: $(6370.01^2 - 6370^2)^{1/2} = 11.3$ km
    - However if we write $(6370 + 0.01)^2 \approx 6370^2 + 2\times0.01\times6370$, the 6370^2 cancels and we get $\sqrt{127.4} = 11.3$ km
    - this is much easier to check than the previous result (you can do $2\times63.7 = 127.4$ in your head, and since $11^2 = 121$, 11.3 must be about right)

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Example 3: Precision

- The highest energy cosmic ray observed by the Auger experiment had a measured energy of 6.4 J. If the particle was a proton, mass 1.7×10^{-27} kg, calculate the difference between its velocity and the speed of light.
  - Relativity gives $E = \gamma mc^2$ where $\gamma = (1 - v^2/c^2)^{-1/2}$
    - so try this: $\gamma = 6.4/(1.7\times10^{-27}\times(3.0\times10^8)^2) = 4.3\times10^{10}$
    - $1 - v^2/c^2 = 1/\gamma^2 = 5.5\times10^{-22}$
    - $v/c = (1 - 5.5\times10^{-22})^{1/2} = 1$ (exactly, on calculator), so $v \equiv c$
    - oops, can't be right, proton is not massless
  - Instead write $v = c(1 - \varepsilon)$ where $\varepsilon << 1$
    - $v^2 = c^2(1 - 2\varepsilon)$ using binomial expansion
    - $\gamma = (2\varepsilon)^{-1/2}$, so $\varepsilon = 0.5/\gamma^2 = 2.7\times10^{-22}$
    - $c - v = c\varepsilon = 8.2\times10^{-14}$ m s^{-1} (!!)

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CHECKS AND ESTIMATES

- Easy to get numerical values wrong
  - mismatch units, e.g. 1.4 mm/184 m gives 7.6×10^{-3} rad instead of 7.6×10^{-6} rad if you forget to convert mm to m
  - think calculator's set to degrees when it's set to radians, or vice versa
  - forget square or square root (or indeed any other power)
  - transpose digits in calculator, e.g. 19 becomes 91
  - forget exponent, or reverse its sign, e.g. 1.6×10^{-19} becomes 1.6×10^{19}
- Therefore should always check before moving on
  - numerical mistakes are usually easy to fix *once you realise you have a problem*

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BASIC CHECKS

- Is the number obviously wrong?
  - satellite orbiting at 6× speed of light, stars in binary system separated by 32 cm, mass of snooker ball = 15 tonnes, etc.
    - Check for forgotten powers
    - Check for incorrect units
- Does the rest of the problem make sense with this number?
  - if you found that the mass of an object was 300 kg, and the question continues, "A student picks up the object" – your number is probably wrong!
- Is the number intrinsically plausible?
  - it is useful to be aware of "typical" values for particular areas of physics or astrophysics
    - e.g. stars typically between 0.1 and 100× Sun's mass

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Estimates

- If at all doubtful, make estimate using powers of 10
  - Example: The Sun has a power output of 3.8×10^{26} W and a radius of 7.0×10^{8} m. What is its surface temperature?
    - We use $L = 4\pi R^2\sigma T^4$ where $\sigma = 5.67\times10^{-8}$ W m^{-2} K^{-4}
    - So we have $T = (L/4\pi R^2\sigma)^{1/4}$
    - Estimate: $4\times10^{26}/(4\times3\times(7\times10^{8})^2\times6\times10^{-8})$
      - powers of 10: 26 – 2×8 + 8 = 18
      - numbers: $4/(4\times3\times50\times6) = 1/900 = 10^{-3}$
      - so $T^4 \sim 10^{15}$ K^4
      - $T = (10^{16}/10)^{1/4} = 10000/(10^{1/4}) \sim 5000$ K (actually 5700 K)
    - Why do this instead of just redoing calculation?
      - it's quicker (with practice)
      - you may be entering calculation into calculator wrongly
      - if you focus on powers, you may find one you forgot!

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Calculations: Summary

- Before entering numbers:
  - check dimensions of equation
  - check behaviour of equation (special cases etc.)
  - think about whether you should be using any expansions or approximations
  - check units of numerical values
- After entering numbers:
  - consider whether result is physically possible
  - consider whether it is plausible
  - if you made any approximations, check whether they were valid
    - e.g. if you assumed $v \ll c$, but the result is $v = 2.8\times10^{8}$ m s^{-1}, you have a problem

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PROBLEM SOLVING: SUMMARY

- Basic approach
  - **Model** the system, and identify the underlying physics
  - **Formulate** equations describing the system
  - **Solve** the equations for the variable(s) you need
  - **Check** that your solution makes sense before using it
- Principles
  - Use symbols, not numbers, for as long as possible
  - Always check dimensions
  - Look for special cases, and check solution works there
  - Check units before substituting numbers
  - Check that numerical answers are
    - physically possible
    - plausible for given system

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AND FINALLY...

PRACTISE!

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