Problem Solving

Dimensional Analysis
- Principles
- Uses
- Estimation

Special Values
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Dimensional Analysis: Principles

- "You can't add apples to oranges"
  - "1 km + 42 kg = 27 s" does not make sense
  - if Z = X + Y, then Z, X, Y must all have same dimensions
- Note distinction between *dimensions* and *units*
  - 1 m, 1 furlong, 1 pc all have *dimensions* of length, but different *units*
    - for symbolic equations only need consistent *dimensions*
    - for numerical calculations also need consistent *units*
  - Example:
    age of universe ≈ $1/H_0$ where H_0 is Hubble's constant
    - H_0 = 71 km s^{-1} Mpc^{-1} has *dimensions* 1/time, as required
    - universe clearly is *not* 1/71 s old!
    - need to convert to appropriate *units* using 1 Mpc = 3.2×10^{19} km

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Basic Dimensions

- SI base units are metre, kilogram, second, ampere, kelvin, mole and candela
  - seldom need the last two in dimensional analysis problems
  - so the 5 main dimensions to consider are length [L], mass [M], time [T] current [I] (or charge [Q]) and temperature [Θ]
    - irritating that both time and temperature begin with "t"...
  - use square brackets to distinguish between *dimensions* and *units*
    - e.g. do not confuse [M] (mass) with unit m (length)

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Uses

- Check correctness of equations
  - "I can't remember if the volume of a sphere is $4\pi R^2$ or $^4/_3\, \pi R^3$"
    - R^2 has dimensions $(\text{length})^2$, whereas volume has dimensions $(\text{length})^3$
    - therefore the first expressions can't possibly be right – it must be the second one
    - note that you can't identify numerical factors by dimensional analysis: $4\pi R^3$ would also look "right"
- Work out functional forms
  - gravitational potential energy of a sphere of mass M and radius R presumably depends on G, M and R
    - M has dimensions of mass, [M], R of length, [L], G is measured in N m^2 kg^{-2} so has dimensions $[L]^3[T]^{-2}[M]^{-1}$
    - energy has dimensions $[M][L]^2[T]^{-2}$ (think of ½ mv^2)
    - so gravitational potential energy must be $\propto GM^2/R$

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Example 1

- The wavelength produced by a wiggler (a component of a free electron laser) has a value that depends on the applied magnetic field, the speed of light and the mass and charge of the electron. Find the functional dependence of the wavelength on these four parameters.
  - Construct equation: $\lambda \propto B^{\alpha}c^{\beta}m^{\gamma}e^{\delta}$
    - we need to find the unknown powers α, β, γ, δ
    - do this by recognising that the dimensions on both sides of the equation must be the same

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Example 1: Answer

- What are the dimensions of magnetic field B?
  - Use $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$: $[F] = [M][L][T]^{-2}$, $[v] = [L][T]^{-1}$, $[q] = [Q]$
  - therefore $[B] = [M][L][T]^{-2} / ([Q][L][T]^{-1}) = [M][Q]^{-1}[T]^{-1}$
- $\lambda \propto B^{\alpha}c^{\beta}m^{\gamma}e^{\delta} \rightarrow [L] = [M]^{\alpha}[Q]^{-\alpha}[T]^{-\alpha}[L]^{\beta}[T]^{-\beta}[M]^{\gamma}[Q]^{\delta}$
  - Equate powers:
    - [L]: $1 = \beta$ (that was easy!)
    - [M]: $0 = \alpha + \gamma$
    - [Q]: $0 = -\alpha + \delta$
    - [T]: $0 = -(\alpha + \beta)$
  - Solve set of simultaneous equations
    - $\beta = 1$, so $\alpha = -1$, $\delta = -1$, $\gamma = 1$
  - $\lambda \propto mc/eB$

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Example 1: Sanity Checks

- Always check your answer!
  - In dimensional-analysis problems obviously can't really check dimensions (circular logic)
  - So check behaviour
- $\lambda \propto mc/eB$
  - recall energy of photon $\propto 1/\lambda$
- What do we logically expect?
  - ☑ no magnetic field → no emission (zero energy)
  - ☑ no charge → no force → no emission
  - ☑ more massive particle → harder to wiggle → lower energy

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Dimensional Analysis: Estimation

- Numerical constants in equations are not usually >10 or <0.1
  - e.g. πr^2, $\frac{1}{2} mv^2$
- Therefore can often obtain an *order of magnitude estimate* by using dimensional analysis and either fundamental constants (e.g. c, h, G, e) or variables that are obviously relevant to the problem (e.g. mass, radius)
  - This can be useful if trying to work out whether an effect is likely to be important or can safely be neglected
    - e.g. "do I need to worry about air resistance here?"

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Example 2

- The radius of an atom is likely to depend on e, ε_0, m_e and h. Estimate the size of atoms.
  - Dimensions of ε_0:
    consider Coulomb's law $F = Q_1Q_2/(4\pi\varepsilon_0 r^2)$
    - ε_0 must have dimensions of $[Q]^2[L]^{-2}[M]^{-1}[L]^{-1}[T]^2 = [Q]^2[L]^{-3}[M]^{-1}[T]^2$
  - $R \propto e^{\alpha}\, \varepsilon_0^{\beta}\, m_e^{\gamma}\, h^{\delta}$
    $[L] = [Q]^{\alpha}\, [Q]^{2\beta}\, [L]^{-3\beta}\, [M]^{-\beta}\, [T]^{2\beta}\, [M]^{\gamma}\, [M]^{\delta}\, [L]^{2\delta}\, [T]^{-\delta}$
  - Equate powers
    - [L]: $1 = -3\beta + 2\delta$
    - [Q]: $0 = \alpha + 2\beta$
    - [M]: $0 = -\beta + \gamma + \delta$
    - [T]: $0 = 2\beta - \delta$

Solution:
$\alpha = -2,\ \beta = 1,\ \gamma = -1,\ \delta = 2$
$R \propto h^2\varepsilon_0/m_e e^2$

  - Numerical value: $R \sim 1.7 \times 10^{-10}$ m
    - the actual value is 0.53×10^{-10} m; we are only a factor of π out

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Special Values

- Check that solution works for simple cases where answer is "obvious"
  - "simple cases" are often extreme values
    - variable goes to 0 or ∞
  - sometimes they can be cases with symmetry
    - two lengths, masses, etc., are equal
- Also check that changing values has expected effect
  - Example:
    - Thermal conductivity, k, has units of W m^{-1} K^{-1}. What is the equation for heat flow P through a barrier of area A and thickness x, if the difference in temperature is ΔT?
    - The three simplest possibilities with correct dimensions are (a) $P = kx\,\Delta T$, (b) $P = k\,\Delta T\,\sqrt{A}$, (c) $P = (kA/x)\,\Delta T$
    - If (a), heat flow <u>increases</u> as barrier gets <u>thicker</u>; if (b), heat flow doesn't depend on thickness at all – so it's got to be (c)

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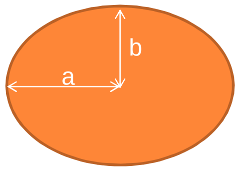
Example 3



- Which of these formulae gives the area of the ellipse?
- (a) ab^2; (b) $a^2 + b^2$; (c) a^3/b; (d) $2ab$ (e) πab
  - only (a) is dimensionally wrong
- Special cases:
  - if $a = 0$, area = 0 (ellipse collapses to vertical line)
  - therefore (b) must be wrong
  - if $b = 0$, area = 0 (ellipse collapses to horizontal line)
  - therefore (c) must be wrong
  - if $a = b$, ellipse becomes a circle with area πa^2 (= πb^2)
  - therefore (d) must be wrong
    and the only possible answer is (e)

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Scaling

- If you know functional dependence and the result for one particular case, you can use ratios to find results for other cases.
  - Example:
    It is known that the luminosity (power output) of a main-sequence star is related to its mass by $L \propto M^{3.8}$.
    Estimate the lifetime of the star Sirius, which is twice the mass of the Sun, if the Sun's lifetime is 10^{10} years.
  - We can reasonably assume that the energy available to the star is proportional to its mass, so $T \propto M/L$
    - dimensional analysis gives Mc^2/L, i.e. $E \propto Mc^2$
  - If $L \propto M^{3.8}$ this gives $T \propto M^{-2.8}$
    - So $T_{Sir}/T_{sun} = (M_{sir}/M_{sun})^{-2.8} = 0.14$, i.e. $T_{sir} = 1.4\times10^9$ yr

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