The Moon’s Phase and the Sun-Earth-Moon geometry

The unilluminated half of the Moon is clearly the half facing away from the Sun. The fraction of this half that we see, i.e. the phase of the Moon, depends on the geometry of the Sun-Earth-Moon triangle as shown here: for example, at full Moon the “triangle” is a straight line Sun-Earth-Moon, whereas at new Moon it is a straight line Sun-Moon-Earth; at first and third quarters it is a right-angled triangle with the right angle at the Moon. Because in fact the Sun is much further away than the Moon, the angle at the Sun is very nearly zero, so the angle between the Sun and Moon as seen from Earth (the Moon’s \textit{elongation}) is very nearly $180^\circ$ minus the angle between the Sun and Earth as seen from the Moon.

You can calculate this angle by measuring the position $E$ of the inner edge of the crescent Moon. As shown in the diagram, the distance between $E$ and the edge of the Moon’s disc is $r - r \cos \theta$, where $\theta$ is defined on the diagram. By looking at the Earth-Sun-Moon triangle, you can see that, if the Sun is infinitely far away, the angle $\theta$ is in fact the Moon’s elongation. Over the course of a lunar month, you should therefore be able to plot the position of the Moon in its orbit relative to the Earth-Sun line as a reference point.