The University Of Sheffield.

Department Of Physics & Astronomy.

Autumn Semester 2005-2006 **2 hours**

MATHEMATICAL METHODS FOR PHYSICISTS AND ASTRONOMERS

Answer question ONE (compulsory) and TWO others.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

TURN OVER

1 COMPULSORY

a) Solve the following differential equations:

i) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -ax$ subject to the boundary condition $x(0) = A$. [1]

ii) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = b^2 y$ subject to the boundary conditions $y(0) = B$ and $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0$. [1]

iii) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -c^2 y$ subject to the boundary condition $y(0) = D$ and $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0$. [1]

b) For each of the three types of equation i), ii) and iii), give one example of its use in physics or astronomy. [3]

c) Draw an Argand diagram showing the complex number $z = 1 + i$, and write z in the form $z = re^{i\phi}$ (finding the values of r and ϕ). [1]

d) Expand the product $(a + be^{i\theta})(a + be^{-i\theta})$, where a and b are real, and simplify the result. [1]

e) Show that when $r \neq 0$ $V(r) = \dfrac{Q}{4\pi\varepsilon_0 r}$ satisfies

$$\nabla^2 V(r) = 0 .$$ [2]

CONTINUED

2 a) Explain what is meant by an even function and an odd function. [1]

If the function $F(x)$ is even and the function $G(x)$ is odd, what is the symmetry of the function $H(x) = F(x)\, G(x)$? [1]

Which of the following integrals are identically zero? Evaluate those that are finite:

$$\int_{-\pi}^{\pi} \mathrm{d}x \cos x \sin x\,,$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{x}{\left(x^2+a^2\right)^2}\,,$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^3 e^{-\alpha x^2}\,,$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \cos\theta\,.$$

[2]

b) Sketch the function given by

$$f\left(\theta\right) = f\left(\theta + 2\pi\right)$$

$$\begin{aligned} f(\theta) &= -1 \quad \mathit{for} \quad -\pi < \theta < 0 \\ &= \ \ 1 \quad \mathit{for} \quad 0 < \theta < \pi\,, \end{aligned}$$

and express this function as a Fourier Series. [6]

TURN OVER

3 a) Show that the Fourier transform of the function

$$f(x) = 1, \qquad |x| < b \text{ and}$$
$$f(x) = 0 \qquad \text{otherwise}$$

is given by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x f(x) e^{ikx} = \frac{1}{\sqrt{2\pi}} \frac{2\sin kb}{k} \; . \qquad [2]$$

Sketch $F(k)$ and $|F(k)|^2$. [2]

b) Two functions $f(x)$ and $g(y)$ are defined by

$$f(x) = 1 \qquad |x| < b \text{ and}$$
$$f(x) = 0 \qquad \text{otherwise and}$$

$$g(y) = \delta(y+a) + \delta(y) + \delta(y-a) \quad \text{where } 2b < a.$$

Sketch the functions $f(x)$ and $g(y)$. [2]

Sketch the function that is the convolution of $f(x)$ and $g(y)$ above. [2]

Using the convolution theorem, or otherwise, evaluate the Fourier Transform of the convolution of $f(x)$ and $g(y)$ above. [2]

CONTINUED

4 The diffusion equation is given by

$$\frac{\partial^2 F(x,t)}{\partial x^2} = \frac{1}{D}\frac{\partial F(x,t)}{\partial t}.$$

Give an example of a physical situation that may be described by this equation. [1]

What are the dimensions of the constant D? [1]

Assume that at $t = 0$ the function $F(x,0)$ is given by $F(x,0) = F_0 + F_1 \sin kx$.

Show that at subsequent times the function $F(x,t)$ is given by

$$F(x,t) = F_0 + F_1 \exp(-t/\tau_k)\sin kx$$

and find an expression for τ_k. [3]

Consider a new starting configuration,

$$F(x,0) = F_0 + F_1\left(\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L}\right),$$

and hence find an expression for $F(x,t)$ at later times. Sketch $F(x,0)$. [3]

Sketch $F(x,t)$ for time $t = \dfrac{L^2}{D\pi^2}$. [2]

5 a) An unbiased coin is tossed 5 times. Find the probability that there is

(i) at least one head, [2]

(ii) at least one head and one tail. [3]

b) The Poisson distribution is defined as $P(n) = \dfrac{\lambda^n e^{-\lambda}}{n!}$.

Show that $\sum_{n=0}^{\infty} P(n) = 1$. [2]

(You may assume that $n! = 1$ for $n = 0$.)

Evaluate $\sum_{n=0}^{\infty} nP(n)$. [3]

END OF QUESTION PAPER