Lectures 5-8: Fourier Series

References *Jordan & Smith* Ch.26, *Boas* Ch.7, *Kreyszig* Ch.11
Some fun 'java applet' demonstrations are available on the web. Try putting 'Fourier series applet' into Google and looking at the sites from jhu, Falstad and Maths Online Gallery.

1. Introduction to Fourier Series

Consider a length of string fixed between rigid supports. The full behaviour of this system can be found by solving a wave equation – a partial differential equation. We will do this later in the course. For now we will just recall the basic properties of waves of strings which we already know:
There is a ***fundamental*** mode of vibration. Call the frequency of this mode ***f*** and the time period ***T***.
Then there are various *harmonics*. These have frequency *2f, 3f, 4f, 5f, …, nf, …*

In practice, when a piano or guitar or other string is hit or plucked, it does not vibrate purely in one mode – the displacement of the string is not purely sinusoidal, the sound emitted is not all of one frequency. In practice, one normally hears a large amount of the fundamental *plus* smaller amounts of various harmonics. The proportions in which the different frequencies are present varies – hence a guitar sounds different from a violin or a piano, and a violin sounds different if it is bowed from if it is plucked!
(See http://www.jhu.edu/~signals/listen/music1.html pages 1&2)

Remember that if the fundamental frequency has frequency f, its period $T = 1/f$.
A harmonic wave of frequency nf will then have a period T/n, but obviously also repeats with period T.
So **if we add together sinusoidal waves of frequency *f, 2f, 3f, 4f,…* the result is a (non-sinusoidal) waveform which is periodic with the same period *T* as the fundamental frequency, *f = 1/T*.**
[E.g. play with http://www.falstad.com/fourier/]

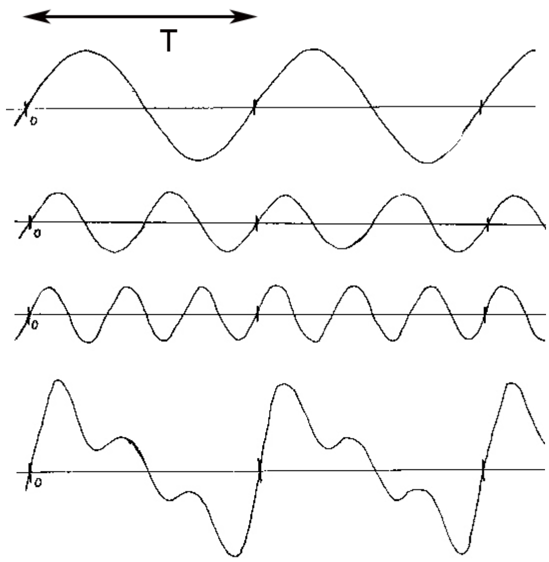
Sometimes we use the angular frequency ω where the nth harmonic has $\omega_n = 2\pi nf = 2\pi n/T$.
The various harmonics are then of the form $A\sin\omega_n t$.

Illustration:

sin ωt

0.5 sin2ωt

0.5 sin3ωt



$y_1(t) = \sin\omega t + 0.5\sin 2\omega t + 0.5\sin 3\omega t$

For all the functions above, the *average value over a period* is zero.
If we add a *constant* term, the waveform remains periodic but its average value is no longer zero:

$y_2(t) = 1 + \sin\omega t + 0.5\sin 2\omega t + 0.5\sin 3\omega t$

What is really *useful* is that this works in reverse:

<u>Any</u> periodic function with period T can be expressed as the sum of a constant term plus harmonic (sine and cosine) curves of angular frequency ω, 2ω, 3ω, ... where $\omega = 2\pi/T$.

i.e. we can write

$$\boxed{\begin{aligned} F(t) &= \frac{1}{2}a_0 + (a_1\cos\omega t + b_1\sin\omega t) + (a_2\cos 2\omega t + b_2\sin 2\omega t) + \ldots \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega t + b_n\sin n\omega t \end{aligned}}$$

where $\omega = 2\pi/T$.

We will later prove this result mathematically, and later in the semester will see that it can be deduced from the general solution of the wave equation. For now you may be able to persuade yourself of its plausibility by playing with the various websites – for example, the demonstrations of how 'square' or 'triangular' waveforms can be made from sums of harmonic waves. The more terms in the sum, the closer the approximation to the desired waveform. Hence in general, an infinite number of terms are needed.

Why is this useful?

In lecture 4 we solved the 'forced harmonic oscillator' equation $\frac{d^2}{dt^2}x(t) + 2\gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = F\cos\omega t$.
Such an equation could describe, for example, the response of an electrical LCR circuit to a sinusoidal driving voltage. But what would happen if we applied a *square wave* driving voltage?? Using Fourier theory, we would just need to express the square waveform as a sum of sinusoidal terms. Then the response would be the sum of the solutions for each term (which would all have similar form, but involve different multiples of ω thus also have different amplitudes). Throughout physics there are many similar situations. Fourier series means that *if we can solve a problem for a sinusoidal function then we can solve it for <u>any</u> periodic function*!

And periodic functions appear everywhere! Examples of periodicity in time: a pulsar, a train of electrical pulses, the temperature variation over 24 hours or the average daily temperature over a year (approximately). Examples of periodicity in space: a crystal lattice, an array of magnetic domains, etc.

Other Forms

If we want to work in terms of *t* not ω, the formula becomes

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{2n\pi t}{T} + b_n\sin\frac{2n\pi t}{T}.$$

Or similarly for a function *f(x)* which is periodic in *space* with repetition length *L*, we have

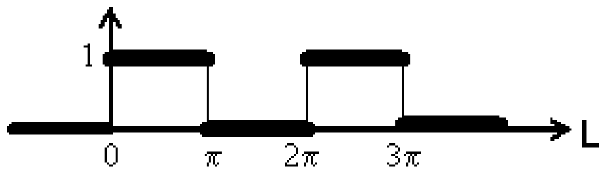
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{2n\pi x}{L} + b_n\sin\frac{2n\pi x}{L}.$$

(Any value of T or L can be used, although to keep the algebra straight forward, most questions will set
T as 2π or even L as 2π metres.)

2. Towards Finding the Fourier Coefficients

To make things easy let's say that the pattern repeats itself every 2π metres, so L = 2π. The Fourier series can then be expressed more simply in the form

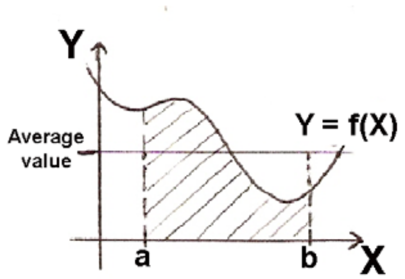
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx.$$



Now we want to find expressions for the coefficients a_n and b_n.
To do this we need two other bits of preparatory mathematics …

(a) Average Value of a Function

Consider a function $y = f(x)$. The *average value* of the function between $x = a$ and $x = b$ is defined to be $\boxed{\frac{1}{b-a}\int_a^b f(x)dx}$.



Geometrically this means that the area under the curve $f(x)$ between a and b is equal to the area of a rectangle of width $(b-a)$ and height equal to this average value.

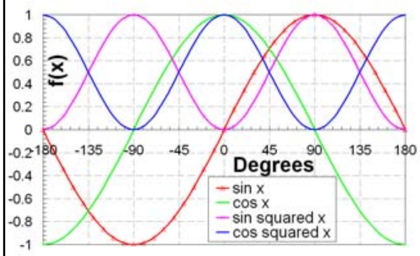
Note that while average values can be found by evaluating the above integral, sometimes they can be identified more quickly from symmetry considerations, a sketch graph and common sense!
Two particularly important results are:

- *The average value of a sine or cosine function over a period is zero*:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos nx\,dx = 0.$$

- *The average value of $\cos^2$ or $\sin^2$ over a period is ½*:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,dx = \frac{1}{2}.$$



Actually both these results can be generalized. It is easily shown that:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos nx\,dx = 0 \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi}\sin^2 nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos^2 nx\,dx = \frac{1}{2} \quad \text{for } n \neq 0$$

Hence $\boxed{\int_0^{2\pi}\sin nx\,dx = \int_0^{2\pi}\cos nx\,dx = 0}$ and $\boxed{\int_0^{2\pi}\sin^2 nx\,dx = \int_0^{2\pi}\cos^2 nx\,dx = \pi}$ $(n \neq 0)$

Note: **We have written all the integrals over [0, 2π] but *any* interval of width 2π can be used**, e.g. [–π, π], [13.1π, 15.1π], etc.

(b) Orthogonality (Proofs in the Appendix)

Sines and cosines have an important property called 'orthogonality':

- *The product of two different sine or cosine functions, integrated over a period, gives zero:*

$$\boxed{\begin{gathered}\int_0^{2\pi}\sin nx\cos mx\,dx = 0 \quad \text{for all } n, m \\ \int_0^{2\pi}\sin nx\sin mx\,dx = \int_0^{2\pi}\cos nx\cos mx\,dx = 0 \quad \text{for all } n \neq m\end{gathered}}$$

Again we can integrate over *any period.*

Equipped with these results we can now find the Fourier coefficients …

3. Fourier Coefficients – Derivation

Earlier we said any function $f(x)$ with period 2π can be written $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$.

Take this equation and integrate both sides over a period (any period):

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0\int_0^{2\pi}dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\,dx + b_n\int_0^{2\pi}\sin nx\,dx\right]$$

Clearly on the RHS the only non-zero term is the a_0 term: $\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0\int_0^{2\pi}dx = \frac{1}{2}a_0(2\pi - 0) = \pi a_0$

hence we find $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$. i.e. $a_0/2$ is the *average value of the function f(x)*.

Now take the original equation again, multiply both sides by cos*x*, then integrate over a period:

$$\int_0^{2\pi} f(x)\cos x dx = \frac{1}{2}a_0\int_0^{2\pi} \cos x dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos x\,dx + b_n\int_0^{2\pi}\sin nx\cos x\,dx\right]$$

On the RHS, this time only the a_1 term survives as it is the only term where n=1 (see Orthogonality.)

$$\int_0^{2\pi} f(x)\cos x dx = a_1\int_0^{2\pi}\cos x\cos x\,dx = a_1\int_0^{2\pi}\cos^2 x\,dx = a_1\pi \qquad \text{hence} \qquad a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx\,.$$

The method for finding the coefficients a_n should thus be clear. To find a general expression for a_n we can take the equation, multiply both sides by cos*mx*, then integrate over a period:

$$\int_0^{2\pi} f(x)\cos mx dx = \frac{1}{2}a_0\int_0^{2\pi} \cos mx dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right]$$

On the RHS, only the a_m term survives the integration:

$$\int_0^{2\pi} f(x)\cos mx dx = a_m\int_0^{2\pi}\cos^2 mx\,dx = a_m\pi \qquad \text{hence} \qquad a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx dx\,.$$

In a similar way, multiplying both sides by sin*mx*, then integrating over a period we get:

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx$$

4. Summary of Results

A function *f(x)* with **period 2π** can be expressed as $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$.

The more general expression from page 2 can be written as:-

A function *f(x)* with **period *L*** can be expressed as $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{2n\pi x}{L} + b_n\sin\frac{2n\pi x}{L}$

where $a_0 = \frac{2}{L}\int_0^{L} f(x)dx$, $a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{2n\pi x}{L}dx$, $b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{2n\pi x}{L}dx$.

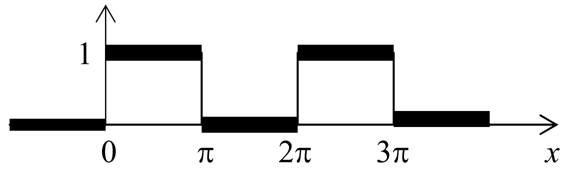
Note

1) The formula for a_0 can be obtained from the formula for a_n just by setting $n = 0$.
2) The integrals above are written over [0, 2π] and [0, *L*] but *any* convenient interval of width one period may be used, and this will be dependent on the nature of the function (see examples and Phil's Problems).
3) The equations can be easily adapted to work with other variables or periodicities. For example, for a function periodic in time with period *T* just replace *x* by *t* and *L* by *T*.
4) A few books use the alternative form $F(t) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n\cos(n\omega_0 t + \theta_n)$ and find values of d_n and θ_n.

5. Examples

Example 1

Find a Fourier series for the square wave shown.



We have $f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ The period is 2π.

Using our formulae for the coefficients we have:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} 1dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0dx = \frac{1}{\pi}[x]_0^{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(0)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$$

So all the a_n coefficients are zero for $n \geq 1$.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(0)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$b_n = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right) = \frac{-1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{1}{n}\right)$$

So when n = odd ; $\cos n\pi = -1$ so $b_{n=odd} = \frac{-1}{\pi}\left(\frac{-1}{n} - \frac{1}{n}\right) = \left(\frac{2}{n\pi}\right)$

So when n = even ; $\cos n\pi = 1$ so $b_{n=even} = \frac{-1}{\pi}\left(\frac{1}{n} - \frac{1}{n}\right) = 0$

The standard Fourier series expression is $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

So the resulting series is: $f(x) = \frac{1}{2} + \frac{2}{\pi}\sin 1x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + ...) = \frac{1}{2} + \frac{2}{\pi}\sum_{n\,odd}\frac{1}{n}\sin nx$

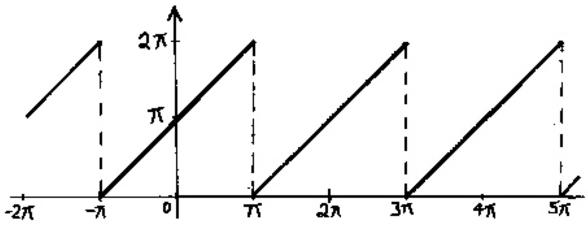
Think: After reaching your answer, ask yourself: *is this result sensible?*

- Does the term $a_0/2$ look like an appropriate value for the average value of the function over a period?
- Would we expect this function to be made mainly of sines or of cosines? (See later for symmetry).
- In what proportions would we expect to find the fundamental and the various harmonics?

(You can also try checking your answer by 'building' the series at http://www.falstad.com/fourier/ or http://www.univie.ac.at/future.media/moe/galerie/fourier/fourier.html)

Example 2

Find a Fourier series of the function shown:



Again the period is 2π.
But this time it is easiest to work with the range [-π, π].
N.B. If we wanted we could use the range [0,2π] and get the same answer, but it would be more fiddly.

Between -π and π, *f(x)* is a straight line with gradient 1 and a Y-intercept of π.
So we can write $f(x) = x + \pi \qquad -\pi < x < \pi$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \pi x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\left\{\frac{\pi^2}{2} + \pi^2\right\} - \left\{\frac{\pi^2}{2} - \pi^2\right\}\right) = \frac{1}{\pi}\left(2\pi^2\right) = 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\cos nx\,dx$$

We must integrate $\int_{-\pi}^{\pi} x\cos nx\,dx$ by parts: $\int u\,dv = uv - \int v\,du$ so set u $= x$ and $\cos nx\,dx = dv$

So $du = dx$ and $v = \int \cos nx dx = \frac{1}{n}\sin nx$.

So $\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{x}{n}\sin nx - \int_{-\pi}^{\pi}\frac{1}{n}\sin nx dx = \left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} = 0$ (see p.3 average value)

Going back to a_n,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\cos nx\,dx = 0 \quad \text{(see p.3 average value)}$$

Now let's find the b_n coefficients….

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\sin nx\,dx$$

We must integrate $\int_{-\pi}^{\pi} x\sin nx\,dx$ by parts: $\int udv = uv - \int vdu$ so set u $= x$ and $\sin nx\ dx = dv$

So $du = dx$ and $v = \int \sin nx dx = -\frac{1}{n}\cos nx$.

So $\int_{-\pi}^{\pi} x\sin nx\,dx = -\frac{x}{n}\cos nx - \int_{-\pi}^{\pi}\frac{-1}{n}\cos nx dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$

Going back to b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\sin nx\,dx = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} + 0$$

$$b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\left(\frac{-\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi\right) - \left(\frac{-(-\pi)}{n}\cos(-n\pi) + \frac{1}{n^2}\sin(-n\pi)\right)\right]$$

Remember that $\cos(-n\pi) = \cos(n\pi)$ and $\sin(-n\pi) = -\sin(n\pi)$

So $b_n = \frac{1}{\pi}\left[\left(\frac{-2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi\right)\right] = -\frac{2}{n}\cos n\pi$. What will b_n be for different values of n?

n = 1	**n = 2**	**n = 3**	**n = 4**
$-\frac{2}{1}\cos 1\pi = -\frac{2}{1}(-1) = \frac{2}{1}$	$-\frac{2}{2}\cos 2\pi = -\frac{2}{2}(1) = -\frac{2}{2}$	$-\frac{2}{3}\cos 3\pi = -\frac{2}{3}(-1) = \frac{2}{3}$	$-\frac{2}{4}\cos 4\pi = -\frac{2}{4}(1) = -\frac{2}{4}$

Hence $f(x) = \frac{2\pi}{2} + 0 + \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + ... = \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

Notes

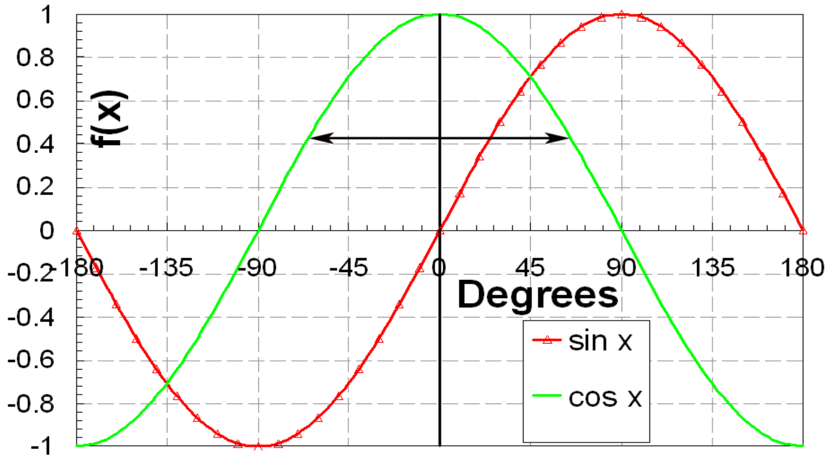
1) Where a function has discontinuities, the Fourier Series converges to the midpoint of the jump (e.g. in example 1 at $x = 0$, π, etc the series has value ½).

2) In general the lowest frequency terms provide the main shape, the higher harmonics add the detail. When functions have discontinuities, more higher harmonics are needed. Hence in both the above examples the terms drop off quite slowly. In general, for smoother functions the terms drop off faster.

6. Even and Odd Functions

For an *even* function, $f_e(-x) = f_e(x)$ i.e. the graph $y = f(x)$ has reflectional symmetry in the y-axis.

For an *odd* function, $f_o(-x) = -f_o(x)$ i.e. the graph $y = f(x)$ has 180º rotational symmetry about the origin.



Any sum of even functions is also an even function.

Hence $\sum_{n=0}^{\infty} a_n \cos nx$ is always an even function.

Therefore **the Fourier series of an even function contains only cosine terms**.

Similarly, **the Fourier series of an odd function contains only sine terms**.

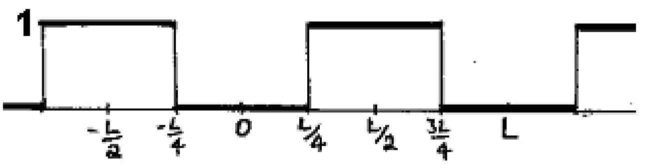
It is **exceptionally useful** to remember this! E.g. if you are asked to find the Fourier series of a function which is even, you can immediately state that $b_n = 0$ for all n, meaning that there will be no sine terms.

You should also remember the following facts (easily verified algebraically or by sketching graphs):

- The product of an *even* function and an *even* function is *even*
- The product of an *odd* function and an *odd* function is *even*
- The product of an *even* function and an *odd* function is *odd*

Example 3

Find a Fourier series of the function shown:



The period is L. As discussed earlier we can integrate over any full period e.g. $\int_0^L$ or $\int_{-L/2}^{L/2}$

The function is *even* and can be written $f(x) = 1$ for $L/4 \le x \le 3L/4$. Therefore there will be no sine terms ($b_n = 0$ for all n) and I feel like integrating between 0 and L. The series will have form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} \quad \text{where} \quad a_0 = \frac{2}{L}\int_0^L f(x)dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx\,.$$

$$\text{So} \quad a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{L}\int_{L/4}^{3L/4} 1dx = \frac{2}{L}[x]_{L/4}^{3L/4} = 1$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx = \frac{2}{L}\int_{L/4}^{3L/4} 1\cos\frac{2n\pi x}{L}dx = \frac{2L}{2n\pi L}\left[\sin\frac{2n\pi x}{L}\right]_{L/4}^{3L/4} = \frac{1}{n\pi}\left\{(\sin\frac{6n\pi}{4}) - (\sin\frac{2n\pi}{4})\right\}$$

$$a_n = \frac{1}{n\pi}\left\{(\sin\frac{6n\pi}{4}) - (\sin\frac{2n\pi}{4})\right\} = \frac{1}{n\pi}\left\{(\sin\frac{3n\pi}{2}) - (\sin\frac{n\pi}{2})\right\}$$

Expression for a_n is not very pretty and easy to make mistakes with. Write out a table to help with assignment of coefficients….

n = 1	**n = 2**	**n = 3**	**n = 4**
$\frac{1}{\pi}\left\{(\sin\frac{3\pi}{2})-(\sin\frac{\pi}{2})\right\}=-\frac{2}{\pi}$	$\frac{1}{2\pi}\left\{(\sin\frac{6\pi}{2})-(\sin\frac{2\pi}{2})\right\}=0$	$\frac{1}{3\pi}\left\{(\sin\frac{9\pi}{2})-(\sin\frac{3\pi}{2})\right\}=\frac{2}{3\pi}$	$\frac{1}{4\pi}\left\{(\sin\frac{12\pi}{2})-(\sin\frac{4\pi}{2})\right\}=0$
n = 5	**n = 6**	**n = 7**	**n = 8**
$\frac{1}{5\pi}\left\{(\sin\frac{15\pi}{2})-(\sin\frac{5\pi}{2})\right\}=-\frac{2}{5\pi}$	$\frac{1}{6\pi}\left\{(\sin\frac{18\pi}{2})-(\sin\frac{6\pi}{2})\right\}=0$	$\frac{1}{7\pi}\left\{(\sin\frac{21\pi}{2})-(\sin\frac{7\pi}{2})\right\}=\frac{2}{7\pi}$	$\frac{1}{8\pi}\left\{(\sin\frac{24\pi}{2})-(\sin\frac{8\pi}{2})\right\}=0$

So $f(x)=\frac{1}{2}-\left(\frac{2}{\pi}\cos\frac{2\pi x}{L}\right)+\left(\frac{2}{3\pi}\cos\frac{6\pi x}{L}\right)-\left(\frac{2}{5\pi}\cos\frac{10\pi x}{L}\right)+\ldots$

7. Half-Range Series

Sometimes we want to find a Fourier series representation of a function which is valid just over some restricted interval. We could do this in the normal way and then state that the function is only valid over a specific interval. However, the fact that we can do this allows us to use a clever trick that reduces the complexity of a problem. We will study this by considering the following example:

Example 4
Consider a guitar string of length L which is being plucked.

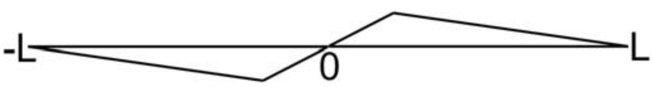
(*Note on application*: If a string was released from this position, finding this Fourier series would be a crucial step in determining the displacement of the string at all subsequent times – see later in course.)

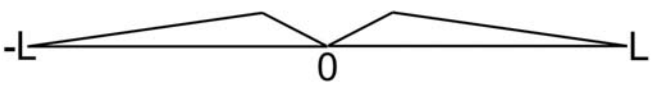
We could, as before, apply the Fourier series to a pretend infinite series of plucked strings and then say that the expression was only valid between 0 and L.

However this series would contain both sine and cosine terms as there is neither even nor odd symmetry, and so would take ages to solve. There is a much more clever way to proceed….

Note that we are only told the form of the function on the interval [0, *L*]. All that matters is that the series corresponds to the given function *in the given interval*. What happens outside the given interval is irrelevant. The way to tackle such a problem is to consider an *artificial function* which coincides with the given function over the given interval, **<u>but extends it and is periodic</u>**. Clearly we could do this in an infinite number of different ways, however in the previous section, we observed that the Fourier series of odd and even functions are particularly simple. It is therefore sensible to choose an odd or even artificial function!

If the original function is defined on the range [0, *L*] then there are always odd and even artificial functions with period 2*L*. In this case these look like:





These functions are called the **odd extension** and **even extension** respectively.
Their corresponding Fourier series are called the **half-range sine series** and **half-range cosine series**.

Theory
We saw earlier that for a function with period *L* the Fourier series is:-

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L},\ \text{where } a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx\,,\ b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

In this case we have a function of period *2L* so the formulae become

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\,,\ \text{where } a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx\,,\ b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Remembering also that $\int_{-b}^{b} f_e(x)dx = 2\int_0^b f_e(x)dx$, we get the following results:

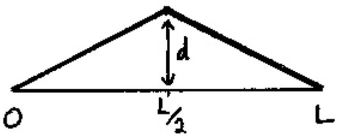
Half-range cosine series: $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$.

Half-range sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$.

Note : The resulting series is only valid over the specified interval!

Example 5

Find a Fourier series which represents the displacement y(x), between x = 0 and L, of the 'plucked string' shown.



Let us choose to find the half-range *sine* series.

We have $y(x) = \begin{cases} 2x\,d/L & 0 < x < L/2 \\ 2(L-x)d/L & L/2 < x < L \end{cases}$

So $b_m = \frac{2}{L}\int_0^L dx \sin\frac{m\pi x}{L} Y(x) = \frac{2}{L}\int_0^{L/2} dx \frac{2dx}{L}\sin\frac{m\pi x}{L} + \frac{2}{L}\int_{L/2}^{L} dx \frac{2d}{L}(L-x)\sin\frac{m\pi x}{L}$

Using integration by parts, it can be shown that the result is: $b_n = \frac{8d}{\pi^2 m^2}\sin\frac{m\pi}{2}$ *for m odd*

$= 0$ *for m even*

So for $0 < x < L$ we have $Y(x) = \frac{8d}{\pi^2}\left[\sin\frac{\pi x}{L} - \frac{1}{9}\sin\frac{3\pi x}{L} + \frac{1}{25}\sin\frac{5\pi x}{L} - \frac{1}{49}\sin\frac{7\pi x}{L} + ...\right]$

Work out the full solution for yourself. This question is answered in "Phil's problems".

8. Further Results

a) Complex Series.

For the waves on strings we need real standing waves. But in some other areas of physics, especially solid state physics, it is more convenient to consider complex or running waves.

Remember that:

$$\cos kx = \frac{1}{2}(e^{ikx} + e^{-ikx}); \qquad \sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx}) = \frac{-i}{2}(e^{ikx} - e^{-ikx})$$

The complex form of the Fourier series can be derived by assuming a solution of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and then by evaluating the coefficients as in section 3, taking the expression and multiplying both sides by e^{-imx} and integrating over a period:

$$\int_0^{2\pi} f(x)e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{inx}e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-m)x}dx$$

For $n \neq m$ the integral vanishes. For $n=m$ the integral gives 2π. Hence $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$

Complex Fourier Series for a function of period **2π**: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$

The more general expression can be written as:-

A function *f(x)* with **period *L*** can be expressed as:- $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$ where $c_n = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}dx$

Let's have a look at an example of complex Fourier series.

Example 6

Find the complex Fourier series for *f(x)* = *x* in the range -2 < *x* < 2 if the repeat period is 4.

$c_n = \frac{1}{L}\int_0^L f(x) e^{-2\pi i n x/L} dx$ and the period is 4. So we can write $c_n = \frac{1}{4}\int_{-2}^{2} x e^{-\pi i n x/2} dx$.

Integration by parts $\int u\, dv = uv - \int v\, du$ with $u = x$ and $dv = e^{-\pi i n x/2} dx$ so $du = dx$ and $v = \frac{-2}{\pi i n} e^{-\pi i n x/2}$

$$c_n = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{-\pi i n x/2} + \int \frac{2}{\pi i n} e^{-\pi i n x/2} dx\right]_{-2}^{2} = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{-\pi i n x/2} - \frac{4}{\pi^2 i^2 n^2} e^{-\pi i n x/2}\right]_{-2}^{2} = \left[\frac{-x}{2\pi i n} e^{-\pi i n x/2} + \frac{1}{\pi^2 n^2} e^{-\pi i n x/2}\right]_{-2}^{2}$$

$$C_n = \left[\frac{-1}{\pi i n} e^{-\pi i n} + \frac{1}{\pi^2 n^2} e^{-\pi i n}\right] - \left[\frac{1}{\pi i n} e^{\pi i n} + \frac{1}{\pi^2 n^2} e^{\pi i n}\right] = \frac{-1}{\pi i n}\left(e^{-\pi i n} + e^{\pi i n}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi i n} - e^{\pi i n}\right)$$

Since $\frac{-1}{i} \times \frac{i}{i} = i$ then $C_n = \frac{i}{\pi n}\left(e^{-\pi i n} + e^{\pi i n}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi i n} - e^{\pi i n}\right)$

It is known that since $e^{\pi i n} = \cos n\pi + i \sin n\pi$ and $e^{-\pi i n} = \cos n\pi - i \sin n\pi$ then ……

$\cos n\pi = \frac{1}{2}\left(e^{-\pi i n} + e^{\pi i n}\right)$ and $\sin n\pi = \frac{-1}{2i}\left(e^{-\pi i n} - e^{\pi i n}\right)$ so we say $C_n = \frac{2i}{\pi n}\cos n\pi - \frac{2i}{\pi^2 n^2}\sin n\pi = \frac{2i}{\pi n}\cos n\pi$

So $C_n = \frac{2i}{\pi n}\cos n\pi = \frac{2i}{\pi n}(-1)^n$ and since $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L}$ then $f(x) = \sum_{n=-\infty}^{\infty} \frac{2i}{\pi n}(-1)^n e^{\pi i n x/2}$

b) Parseval's Theorem

Consider again the Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

Square both sides then integrate over a period: $\int_0^{2\pi} [f(x)]^2 dx = \int_0^{2\pi} \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right]^2 dx$

The RHS will give both squared terms and cross term. When we integrate, *all* the cross terms will vanish. All the squares of the cosines and sines integrate to give π (half the period). Hence

$$\boxed{\int_0^{2\pi} [f(x)]^2 dx = 2\pi \frac{a_0^2}{4} + \pi \sum_{n=1}^{\infty} [a_n^2 + b_n^2]}$$

The energy in a vibrating string or an electrical signal is proportional to an integral like $\int_0^{2\pi} [f(x)]^2 dx$.

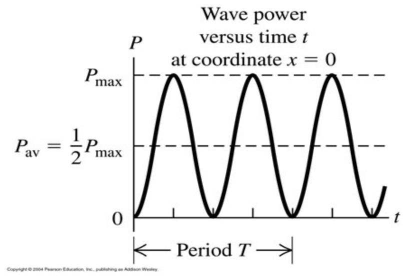
Hence Parseval's theorem tells us that the total energy in a vibrating system is equal to the sum of the energies in the individual modes.

Taken from PHY102

Sinusoidal Wave on a string

$$P_{\max} = \sqrt{\mu T}\,\omega^2 A^2$$

$$P_{ave} = \frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2$$



NB, for all MECHANICAL waves, $P_{ave} \propto A^2\omega^2$

Appendix: Orthogonality

At a fundamental mathematical level, the reason the Fourier series works – the reason any periodic function can be expressed as a sum of sine and cosine functions – is that sines and cosines are orthogonal.

In general, a set of functions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ is said to be *orthogonal* on the interval [*a, b*] if

$$\int_a^b u_n(x)u_m(x)\,dx = \begin{cases} 0 & n \neq m \\ c_n & n = m \end{cases}$$ (where c_n is a constant).

Here we will prove that function sin*nx*, cos*mx*, etc are orthogonal on the interval [0, 2π].

1. $\int_0^{2\pi} \sin nx \cos mx\,dx = \frac{1}{2}\int_0^{2\pi} \sin(n+m)x - \sin(n-m)x\,dx$ [Using $\sin(a+b)-\sin(a-b) = 2\sin a\cos b$]

$$= \frac{1}{2}\left[-\frac{1}{n+m}\cos(n+m)x + \frac{1}{n-m}\cos(n-m)x\right]_0^{2\pi} = 0$$

Hence $\boxed{\int_0^{2\pi} \sin nx \cos mx\,dx = 0 \text{ for } n \neq m.}$

2. $\int_0^{2\pi} \sin nx \sin mx\,dx = \frac{1}{2}\int_0^{2\pi} \cos(n-m)x - \cos(n+m)x\,dx$ [Using $\cos(a-b)-\cos(a+b) = 2\sin a\sin b$]

$$= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n-m}\sin(n-m)x\right]_0^{2\pi} = 0$$

Hence $\boxed{\int_0^{2\pi} \sin nx \sin mx\,dx = 0 \text{ for } n \neq m.}$

3. $\int_0^{2\pi} \cos nx \cos mx\,dx = \frac{1}{2}\int_0^{2\pi} \cos(n+m)x + \cos(n-m)x\,dx$ [Using $\cos(a-b)+\cos(a+b) = 2\cos a\cos b$]

$$= \frac{1}{2}\left[\frac{1}{n+m}\sin(n+m)x + \frac{1}{n-m}\sin(n-m)x\right]_0^{2\pi} = 0$$

Hence $\boxed{\int_0^{2\pi} \cos nx \cos mx\,dx = 0 \text{ for } n \neq m.}$

For $n = m \neq 0$ the integrals becomes:

1. $\int_0^{2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_0^{2\pi} \sin 2nx\,dx = \left[-\frac{1}{4n}\cos 2nx\right]_0^{2\pi} = 0$
2. $\int_0^{2\pi} \sin^2 nx\,dx = \frac{1}{2}\int_0^{2\pi} (1-\cos 2nx)\,dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_0^{2\pi} = \pi$
3. $\int_0^{2\pi} \cos^2 nx\,dx = \frac{1}{2}\int_0^{2\pi} (1+\cos 2nx)\,dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin 2nx\right]_0^{2\pi} = \pi$

For $n = m = 0$ the first two integrals become $\int_0^{2\pi} 0\,dx = 0$ and the third becomes $\int_0^{2\pi} 1\,dx = 2\pi$

<u>Note</u>

1. Similar results can be proved for function of periodicity *L*.
2. The results ($n \neq 0$) are easy to remember: ALL integrals over sines and cosines over a full period give zero, unless the integrand is a square in which case the integral is always equal to half the range of the integral.