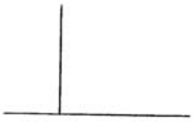
Lecture 12: Convolution Integrals

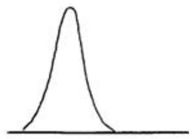
References *Jordan & Smith* 27.7, *Boas* Ch.15.4, *Kreyszig* 11.9.
Web site http://www.jhu.edu/~signals/ : go to 'The joy of convolutions'

1. Convolution in measurements

Assume we have an absolutely sharp line or image. Our measuring system will always render a signal that is 'instrumentally limited'; this is often called the 'resolution function'.

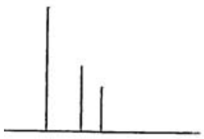


true signal

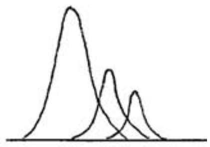


what we detect

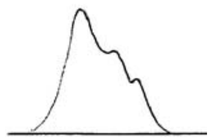
Now suppose we have a composite signal. Every bit of it will give rise to a broad line as above.



true signal



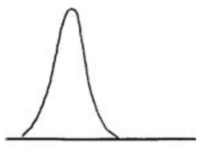
all three lines broaden



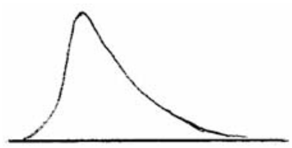
what we detect

If the true signal is itself a broad line then what we detect will be a *convolution* of the signal with the resolution function:

true signal, *f*



resolution function, *g*

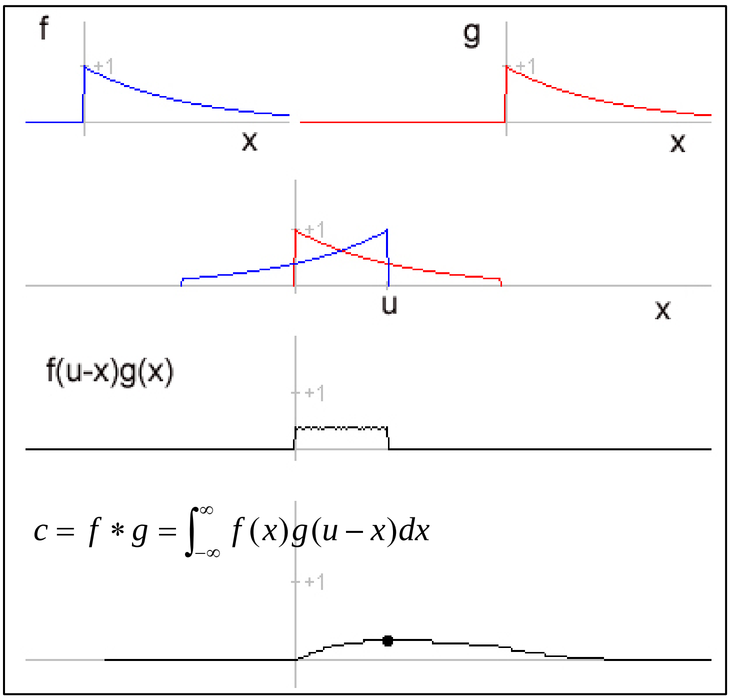


what we detect, *c*

We see that the convolution is broader then either of the starting functions. Convolutions are involved in almost all measurements. If the resolution function g(t) is similar to the true signal f(t), the output function c(t) can effectively mask the true signal. Convolutions are best explored using the jhu website: http://www.jhu.edu/~signals/convolve/index.html . The figure below has been adapted from the website.

2. Theory

The actual way that the functions combine is by definition convoluted. The process is as follows:-



1. Let both functions be given in terms of *x*.
2. Reflect the true signal function: $f(x) \rightarrow f(-x)$.
3. Add an offset, *u*, which allows $f(u-x)$ to move along the x-axis.
4. Start *u* at $-\infty$ and move to $+\infty$. Where the two functions intersect, find the product of both functions.
5. For each value of u take the integral of this product, effectively measuring the area, recording this as *c(x)*. The resulting waveform is the convolution of functions *f* and *g*.

A convolution function is therefore defined by

$$c = f * g = \int_{-\infty}^{\infty} f(x) g(u-x) dx.$$

This process can only really be performed using computers.

3. Deconvolution

We have a problem! We can measure the resolution function (by studying what we believe to be a point source or a sharp line. We can measure the convolution. What we *want* to know is the true signal! This happens so often that there is a word for it – we want to 'deconvolve' our signal.

There is however an important result called the 'Convolution Theorem' which allows us to gain an insight into the convolution process. Let the Fourier transform of the convolution be $C(k)$. Then the convolution theorem states that:-

$$\boxed{C(k) = \sqrt{2\pi}F(k)G(k)}.$$

i.e. *the FT of a convolution is the product of the FTs of the original functions.*

We therefore find the FT of the observed signal, *c(x)*, and of the resolution function, *g(x)*, and use the result that that $C(k) = \sqrt{2\pi}F(k)G(k)$ in order to find *f(x)*.

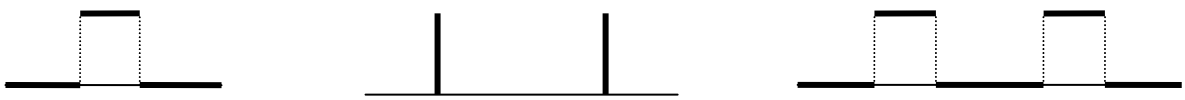
We have $F(k) = \dfrac{C(k)}{G(k)\sqrt{2\pi}}$. So taking the inverse transform, $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{ikx}\frac{C(k)}{G(k)}dk$.

This is not easy and we need lots of signal processing tricks to do it. But it can be and is done in state of the art experiments and experimental observations.

4. Application to Optics

There is a particular use of the convolution theorem here. We have said (and it will be proved in PHY227) that the far field (Fraunhofer) diffraction pattern is the modulus squared of the Fourier transform of the scattering function. If the scattering function is a convolution of two functions then the observed scattering will be the *product* of the scatterings expected from each of the two functions. Here we convolve a top hat function with 2 delta functions so as to yield the characteristic equation representative of light intensity produced from infinitely narrow slits.

Example: Double slits
Consider two delta functions convolved with the single slit 'top hat' function:



single 'top hat' function $f(x) = 1 \quad -a < x < a$

two delta functions $g(x) = \delta(x+d) + \delta(x-d)$

convolution $c(x) = 1$ for $|x-d|<a$ and $|x+d|<a$

Let us find the Fourier transforms of f and g, and their modulus squared.
From earlier, we know that the FT of a top hat function is a sinc function:

$$F(k) = \frac{2}{\sqrt{2\pi}}\frac{\sin ka}{k} \quad \text{so} \quad |F(k)|^2 = \frac{2}{\pi}\frac{\sin^2 ka}{k^2}$$

For the two delta functions we have

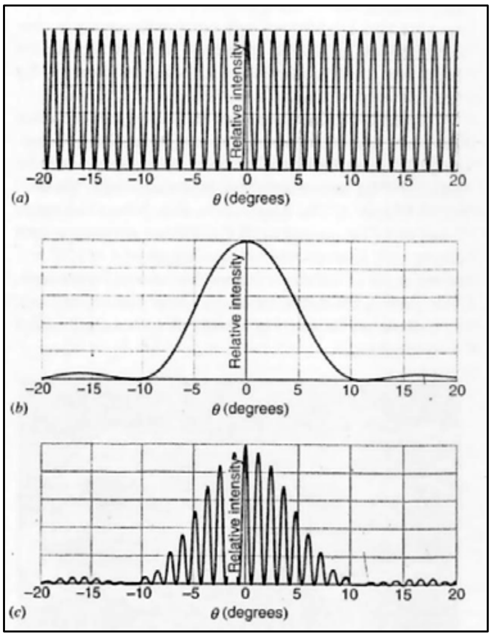
$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\delta(x+d) + \delta(x-d)]e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}[e^{ikd} + e^{-ikd}] = \frac{2}{\sqrt{2\pi}}\cos kd . \quad \text{So} \quad |G(k)|^2 = \frac{2}{\pi}\cos^2 kd .$$

These are the familiar $\cos^2$ fringes – as we expect for two 'infinitely narrow' slits.

The diffraction pattern observed for the double slits will be the modulus squared of the Fourier transform of the whole diffracting aperture, i.e. the convolution of the delta functions with the top hat function.

Hence $|C(k)|^2 = 2\pi|F(k)|^2|G(k)|^2 = \dfrac{4}{\pi^2}\dfrac{\sin^2 ka}{k^2}\cos^2 kd$

– We see $\cos^2$ fringes modulated by a sinc^2 envelope as expected.



$I = I_m \cos^2 kd$ (a) Double slits, separation *d*, infinitely narrow

$I = I_m \left(\frac{\sin ka}{ka}\right)^2$ (b) A single slit of finite width a

$I = I_m \cos^2 kd \left(\frac{\sin ka}{ka}\right)^2$ (c) Double slits of separation *d*, width *a*

5. Convolving Two Gaussians

Here we prove a result about the convolution of two Gaussians with widths related to *a* and *b*. Doing convolution integrals can be difficult but multiplying two FT's is easy.

Earlier we found that Gaussian $f(x) = \sqrt{\frac{a}{2\pi}} e^{-ax^2/2}$ has a FT which is also a Gaussian $F(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2a}}$.

Let us now consider the convolution of two Gaussians $f_a(x) = \sqrt{\frac{a}{2\pi}} e^{-ax^2/2}$ and $f_b(x) = \sqrt{\frac{b}{2\pi}} e^{-bx^2/2}$.

From before we know that $F_a(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2a}}$ and $G_b(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2b}}$.

Hence we can immediately write down $C(k)$:

$$C(k) = \sqrt{2\pi} F_a(k) G_b(k) = \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2a}} \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2b}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}(\frac{1}{a}+\frac{1}{b})} = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2\Delta}} \quad \text{where } \frac{1}{\Delta} = \frac{1}{a} + \frac{1}{b}.$$

This is the just the FT of a single Gaussian $c(x) = \sqrt{\frac{\Delta}{\pi}} e^{-\Delta x^2/2}$ characterised by Δ.

This is an important result: we have shown that the convolution of two Gaussians characterised by *a* and *b* is also a Gaussian and is characterised by Δ. The value of Δ is dominated by whichever is the *smaller* of *a* or *b*, and is always smaller then either of them. Since we found that the width of the Gaussian *f(x)* is $\sqrt{8/a}$ in lecture 9, it is the *widest* Gaussian that dominates, and the convolution of two Gaussians is always wider than either of the two starting Gaussians. From our diagrams we should have been expecting this.

Depending on the experiment, physicists and especially astronomers sometimes assume that both the detected signal and the resolution functions are Gaussians and use this relation in order to estimate the true width of their signal.

6. Proof of the Convolution Theorem (*Optional*)

In section 1 we defined a convolution. Now take the FT of this:

$$C(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(x-u)du\, e^{-ikx}dx$$

Now change the order of integration, then introduce a new variable $v = x - u$ and write $ikx = ik(v+u)$:

$$C(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(x-u)du\, e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du f(u)e^{-iku}\int_{-\infty}^{\infty} g(v)e^{-ikv}dv = \sqrt{2\pi}F(k)G(k).$$

We can use the convolution theorem to prove Parseval's formula for Fourier Transforms.

7. Parseval's Formula

This is analogous to the result discussed for Fourier series. It is important as it relates the total signals integrated over either set of variables.

Relating *f(x)* and *F(k),* $\boxed{\int_{-\infty}^{\infty} dk\,|F(k)|^2 = \int_{-\infty}^{\infty} dx\,|f(x)|^2}$ or *F(ω)* and *f(t),* $\boxed{\int_{-\infty}^{\infty} d\omega\,|F(\omega)|^2 = \int_{-\infty}^{\infty} dt\,|f(t)|^2}$

For example, when are considering Fraunhofer diffraction, for *f(x)* and *F(k)* this formula means the total amount of light forming a diffraction pattern on the screen is equal to the total amount of light passing through the aperture; or for *F(ω)* and *f(t)*, the total amount of light that is recorded by the spectrometer (dispersed according to frequency) is equal to the total amount of light that entered the detector in that time interval.

Proof (optional)

A convolution has form $c(x) = f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du$.

Let us choose to make a convolution of *f* with itself. And since the definition is true for all *x* including *x* = 0 we are free to put *x* = 0: $c(0) = \int_{-\infty}^{\infty} du f(u) f(-u) = \int_{-\infty}^{\infty} du\,|f(u)|^2$. (1)

Using the convolution theorem we know that $C(k) = \sqrt{2\pi}F(k)F(-k) = \sqrt{2\pi}\,|F(k)|^2$.

We back transform *C*(*k*): $c(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C(k)e^{ikx}dk$

And again put *x* = 0: $c(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk C(k) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\,|F(k)|^2 = \int_{-\infty}^{\infty} dk\,|F(k)|^2$. (2)

Equating the two expressions (1) and (2) for *c*(0), we have proved Parseval's theorem.